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A BRIEF PROOF OF CAUCHY'S INTEGRAL THEOREM

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ABSTRACT. A short proof of Cauchy's theorem for circuits homologous to 0 is presented. The proof uses elementary local properties of analytic functions but no additional geometric or topological arguments.

The object of this note is to present a very short and transparent proof of Cauchy's theorem for circuits homologous to 0. The proof is based on simple 'local' properties of analytic functions that can be derived from Cauchy's theorem for analytic functions on a disc, and it may be compared with the treatment in Ahlfors [1, pp. 137–145]. It is apparent from this proof that this version of Cauchy's theorem is not only much more natural than the homotopic version which appears in several recent textbooks; it is also much easier to prove (contra Dieudonné [2, p. 192]). It is reasonable to argue that the concept of homotopy in connection with Cauchy's theorem is as extraneous as the notion of Jordan curve.

We recall that if γ is a circuit (="continuous, piecewise smooth, closed curve"), and $w \in \mathbf{C}$ does not lie on γ, then the *index* of w with respect to γ is $\operatorname{Ind}(\gamma, w) = (2\pi i)^{-1}\int_\gamma (z-w)^{-1}dz$. It is easily proved that $E = \{w \in \mathbf{C} \mid \operatorname{Ind}(\gamma, w) = 0\}$ contains a neighbourhood of ∞ and is open (see [1, p. 116]). In the following proof we give full references to the 'local' properties used in order to emphasize the elementary nature of the proof.

CAUCHY'S THEOREM. *Let D be an open subset of $\mathbf{C}$ and let γ be a circuit in D. Suppose that γ is homologous to 0 in D, i.e. each $w \notin D$ lies in the set E defined above. Then, for each f analytic on D:*

(i) $\int_\gamma f(z)dz = 0$;

(ii) $\operatorname{Ind}(\gamma, w)f(w) = (2\pi i)^{-1}\int_\gamma (z-w)^{-1}f(z)dz$ *for all* $w \in D$ *not lying on* γ.

PROOF. Consider $g: D \times D \to \mathbf{C}$ defined by $g(w, z) = (f(z) - f(w))/(z-w)$ for $z \neq w$ and $g(w, w) = f'(w)$. Then g is continuous, and for each fixed z, $w \mapsto g(w, z)$ is analytic [1, p. 124]. Define $h: \mathbf{C} \to \mathbf{C}$ by $h(w)$

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$=\int_\gamma g(w, z)dz$ on D and $h(w)=\int_\gamma (z-w)^{-1}f(z)dz$ on E. Note that $\boldsymbol{C}=D\cup E$ by hypothesis, and that these two expressions for $h(w)$ are equal on $D\cap E$ because $\mathrm{Ind}(\gamma, w)=0$ there.

Now h is differentiable on both D and E ([1, p. 123] or [3, p. 137]), and so h is an entire function. Since the image of γ is bounded, and E contains a neighbourhood of ∞, $h(w)\to 0$ as $w\to\infty$. This implies firstly that h is constant (Liouville's theorem), and secondly that h is 0. Thus $\int_\gamma g(w, z)dz=0$ for all $w\in D$ not lying on γ; and (ii) follows. Finally, let u be some fixed point of D not lying on γ. Then applying (ii) to the function $z\mapsto f(z)(z-u)$ in place of f, and evaluating at $w=u$, we obtain (i).

REMARK. The proof goes through word for word when γ is a cycle (see [1, p. 138]) rather than a circuit. Then, as in Ahlfors' treatment, the general form of the residue theorem follows immediately.

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