Non-Book Problem #1) We seek solutions to the equation $z^5 = 3 + 3i = 3\sqrt{2}e^{i\pi/4}$. Clearly, the absolute value of any such z must be

$$|z| = \sqrt[10]{27 * 2} = \sqrt[10]{54}.$$

Furthermore, the argument of such a z must satisfy

$$\arg(z) * 5 = \frac{\pi}{4} \pm 2\pi k \qquad \text{where } k \in \mathbb{Z}.$$

Thus, the possible values of $\arg(z)$ within $[0, 2\pi)$ are $\pi/20, 5\pi/20, 9\pi/20, 13\pi/20$, and $17\pi/20$. Thus, the fifth roots of $3 + 3i$ are

$$\sqrt[10]{54}e^{\pi i/20},\ \sqrt[10]{54}e^{5\pi i/20},\ \sqrt[10]{54}e^{9\pi i/20},\ \sqrt[10]{54}e^{13\pi i/20}, \text{ and } \sqrt[10]{54}e^{17\pi i/20}.$$

1.1.4

3) First, we do some basic manipulation, using the assumption that we do not have $|a| = |b| = 1$, meaning $1 - \overline{a}b \neq 0$:

$$\begin{aligned}
&\left|\frac{a-b}{1-\overline{a}b}\right| = 1 \\
\Leftrightarrow &\left|\frac{a-b}{1-\overline{a}b}\right|^2 = 1 \\
\Leftrightarrow &(a-b)\overline{(a-b)} = (1-\overline{a}b)\overline{(1-\overline{a}b)} \\
\Leftrightarrow &|a| + |b| - \overline{a}b - \overline{b}a = 1 + |a||b| - \overline{a}b - \overline{b}a \\
\Leftrightarrow &|a| + |b| = 1 + |a||b|.
\end{aligned}$$

Thus, if either $|a|$ or $|b|$ is 1, then the above expression is trivially true. Furthermore, if $|a| = |b| = 1$, then as long as $a \neq b$, we will have $1 - \overline{a}b \neq 0$, meaning the above manipulation is still valid. We can conclude that in all cases *except* $a = b$ and $|a| = 1$, if $|a| = 1$ or $|b| = 1$ we may say

$$\left|\frac{a-b}{1-\overline{a}b}\right| = 1.$$

4) For this problem, let $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$, and $z = z_1 + iz_2$. The equation

$$az + b\overline{z} = -c$$

gives us a system of two equations when we examine the real and imaginary parts separately:

$$\begin{aligned}
(a_1 + b_1)z_1 + (b_2 - a_2)z_2 &= -c_1 \\
(a_2 + b_2)z_1 + (a_1 - b_1)z_2 &= -c_2.
\end{aligned}$$

Thus, we will have a unique complex solution $z = z_1 + iz_2$ if and only if the above system is consistent and independent, that is, if and only if

$$\det\left(\begin{bmatrix} a_1 + b_1 & b_2 - a_2 \\ a_2 + b_2 & a_1 - b_1 \end{bmatrix}\right) \neq 0.$$

As the determinant of the above matrix is

$$a_1^2 + a_2^2 - b_1^2 - b_2^2 = |a|^2 - |b|^2,$$

we conclude that the equation $az + b\overline{z} + c = 0$ will have exactly one solution whenever $|a| \neq |b|$.

To solve for z_1 and z_2, we simply apply Cramer's rule:

$$z_1 = \frac{-c_1(a_1 - b_1) + c_2(b_2 - a_2)}{|a|^2 - |b|^2}$$

$$z_2 = \frac{-c_2(a_1 + b_1) + c_1(a_2 + b_2)}{|a|^2 - |b|^2}.$$

1.1.5

1) Using the same algebra as in 1.1.4 # 3 (but replacing = with <), we can see that

$$\left|\frac{a-b}{1-\overline{a}b}\right| < 1 \Leftrightarrow |a|^2 + |b|^2 < 1 + |a|^2|b|^2.$$

Note that we can ignore the case where $\overline{a}b = 1$, as by assumption $|a| < 1$ and $|b| < 1$. We can rewrite the above expression as

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$$|a|^2 + |b|^2 - |a|^2|b|^2 < 1 \Leftrightarrow |a|^2(1 - |b|^2) + |b|^2 < 1.$$

However, since $|b| < 1$, $1 - |b|^2 > 0$, and since $|a| < 1$, we may say $|a|^2(1 - |b|^2) < (1 - |b|^2)$. Thus, in our situation,

$$|a|^2(1 - |b|^2) + |b|^2 < (1 - |b|^2) + |b|^2 = 1,$$

as desired.

4) For this problem, we freely use the fact that $|z - a| = d(z, a)$ in the complex plane. By the triangle inequality, we know that

$$d(z, a) + d(z, -a) \geq d(-a, a) \Rightarrow |z - a| + |z + a| \geq 2|a|.$$

By the above expression, it follows that, for this equation to have a solution, it must be that $|a| \leq |c|$, as otherwise the expression would violate the triangle inequality. By similar geometric logic, if $|a| \leq |c|$, then there is at least one point (usually 2) in $\mathbb{C}$ with distance $|c|$ from both a and $-a$. Any such point can be a solution to this equation.

Observe that if $|a| = |c|$, then the only solution to this equation is to have $z = 0$, so it is possible for $|z| = 0$. On the other hand, the triangle inequality also tells us that $|z| \leq |c|$. Since setting $a = 0$ allows $z = c$ as a solution, we conclude that $0 \leq |z| \leq |c|$.

1.2.2

5) As long as $n \nmid h$,

$$1 - \omega^h + \omega^2 h - \cdots + (-1)^{n-1}\omega^{(n-1)h} = \sum_{k=0}^{n-1} (-\omega^h)^k$$

is a finite geometric series with more than one summand. Thus,

$$\sum_{k=0}^{n-1} (-\omega^h)^k = \frac{1-(-\omega^h)^n}{1+\omega^h} = \frac{1-(-1)^n}{1+\omega^h} = \begin{cases} 0 & \text{if} \quad n \in 2\mathbb{Z} \\ \frac{2}{1+\omega^h} & \text{if} \quad n \in 2\mathbb{Z}+1 \end{cases}.$$

1.2.3

1) By our work in problem 1.1.4 #4, we know that it is sufficient to study the following system of equations (derived by studying the real and imaginary parts separately):

$$(a_1 + b_1)z_1 + (b_2 - a_2)z_2 = -c_1$$
$$(a_2 + b_2)z_1 + (a_1 - b_1)z_2 = -c_2.$$

This will have exactly one solution when the system is independent and consistent (which occurs when $|a| \neq |b|$, and it will have zero solutions when the system is inconsistent. Thus, we need to study the situation where $|a| = |b|$, and where not all of a, b, and c are zero (since then the entire plane is a solution). If we insist on those conditions, then the system will have a line as a solution when there exists a constant $k \in \mathbb{R}$ such that

$$c_1 = kc_2$$
$$a_1 + b_1 = k(a_2 + b_2)$$
$$b_2 - a_2 = k(a_1 - b_1).$$

2) For this problem, we use that absolute value is a distance function, and apeal to the geometric definitions of ellipse, hyperbola, and parabola. Let f_1 and f_2 be two distinct points in $\mathbb{C}$, and let $c \in \mathbb{C}$ such that $|c| > |f_1 - f_2|$. For any such c, An ellipse with foci at f_1 and f_2 will be given by the points z satisfying

$$|z - f_1| + |z - f_2| = |c|.$$

Similarly, for any $c \in \mathbb{C}$, the following equation will give a hyperbola:

$$|z - f_1| - |z - f_2| = \pm|c|.$$

Note that this allows some degenerate cases.
Finally, for a parabola, let $a, b, f \in C$ where $a \neq 0$. The following equation will give a parabola:

$$|f - z| = \min_{t \in \mathbb{R}} |z - (a + bt)|.$$

Here, f is the focus and $a + bt$ is the directrix of the parabola.

1.2.4

1) ($\Rightarrow$) Assume $z\overline{z'} = -1$. Note that this implies $\overline{z}z' = -1$. z and z' will correspond to anti polar points on the sphere if the distance between those points (in $\mathbb{R}^3$) is maximized, that is, when the distance is 2. Using the distance equation from class, we get that

$$\begin{aligned} d(z,z') &= \frac{2|z-z'|}{\sqrt{(1+|z|^2)(1+|z'|^2)}} \\ \Leftrightarrow d(z,z')^2 &= \frac{4(z-z')\overline{(z-z')}}{|z|^2+|z'|^2+1+|z|^2|z'|^2} \\ &= 4\frac{|z|^2+|z'|^2-z\overline{z'}-\overline{z}z'}{|z|^2+|z'|^2+1+(z\overline{z'})(\overline{z}z')} \\ &= 4\frac{|z|^2+|z'|^2+2}{|z|^2+|z'|^2+2} = 4 \Leftrightarrow \quad d(z,z') = 2 \end{aligned}$$

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Thus, the assumption that $z\overline{z'} = -1$ implies that the chordal distance between the corresponding points on the Riemann sphere is 2, so these points must be antipodal.

($\Leftarrow$) Assume z and z' correspond to antipolar points on the sphere. Since $\mathbb{C}$ is a field, there is a unique number $x \in \mathbb{C}$ satisfying $zx = -1$. By part (a), x and z correspond to antipolar points. However, since each point on the sphere has exactly one antipolar point, it must be that $x = z'$.

2) Observe that our cube must have vertices at

$$\{c * (x_1, x_2, x_3) : x_1, x_2, x_3 \in \{-1, 1\}\}$$

for some nonnegative, real constant c. By elementary geometry, we may see that $c = 1/\sqrt(3)$ suffices. By the formula for a stereographic projection, we find the corresponding points in $\mathbb{C}$ are

$$\left\{\frac{x+iy}{1-z} : x, y, z \in \left\{\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}\right\}.$$

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