All the exercises are obtained from *Complex Analysis*, Third Edition, by Lars Ahlfors.

1. (Exercise 1, Section 4.2.3) Compute

   (a) $\int_{|z|=1} e^z z^{-n}\, dz$;

   (b) $\int_{|z|=2} z^n (1-z)^m\, dz$;

   (c) $\int_{|z|=\rho} |z-a|^{-4}\, |dz|$, where $|a| \neq \rho$.

Solution.

(a) Since $|z| = 1$ is a circle containing 0, we know that

$$\begin{aligned}\int_{|z|=1} \frac{e^z}{z^n}\, dz &= \frac{2\pi i}{(n-1)!}\left(\frac{(n-1)!}{2\pi i}\int_{|z|=1} \frac{e^z}{(z-0)^n}\, dz\right)\\ &= \frac{2\pi i}{(n-1)!}\, \frac{d^{n-1}}{dz^{n-1}}(e^z)\bigg|_{z=0}\\ &= \frac{2\pi i}{(n-1)!}.\end{aligned}$$

We have implicitly assumed that $n > 0$. If $n \leq 0$, then the exercise is trivial for $e^z z^{-n}$ would be an analytic function —so the integral would evaluate to 0.

(b) We consider two separate cases. We do not treat the case when $n, m \geq 0$ for the integrand $z^n(1-z)^m$ is analytic in the inside of $|z| = 2$, so the value of the integral is 0.

   i. Case $n < 0, m \geq 0$: We see that

$$\begin{aligned}\int_{|z|=2} \frac{(1-z)^m}{z^n}\, dz &= \frac{(n-1)!}{2\pi i}\left(\frac{2\pi i}{(n-1)!}\int_{|z|=1} \frac{(1-z)^m}{(z-0)^n}\, dz\right)\\ &= \frac{2\pi i}{(n-1)!}\, \frac{d^{n-1}}{dz^{n-1}}(1-z)^m\bigg|_{z=0}\\ &= \frac{2\pi i}{(n-1)!}\, \frac{d^{n-1}}{dz^{n-1}}\sum_{r=0}^{m}\binom{m}{r}(-1)^r z^r\bigg|_{z=0}\\ &= \begin{cases} 0, & \text{if } m > n-1,\\ (-1)^{n-1}\dfrac{2\pi i}{(n-1)!}\dbinom{m}{n-1}(n-1)(n-2)\cdots 2\cdot 1, & \text{if } m \leq n-1,\end{cases}\end{aligned}$$

or, in other words,

$$\int_{|z|=2} \frac{(1-z)^m}{z^n}\,dz = \begin{cases} 0, & \text{if } m > n-1, \\ (-1)^{n-1} 2\pi i \binom{m}{n-1}, & \text{if } m \le n-1, \end{cases}$$

ii. Case $n \ge 0, m < 0$: Very similar to the previous case. We have

$$\begin{aligned} \int_{|z|=2} \frac{z^n}{(1-z)^m}\,dz &= \frac{2\pi i}{(-1)^m (m-1)!}\left(\frac{(m-1)!}{2\pi i}\int_{|z|=2}\frac{z^n}{(z-1)^m}\,dz\right) \\ &= (-1)^m \frac{2\pi i}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}} z^n\Big|_{z=1} \\ &= \begin{cases} 0, & \text{if } m-1 > n; \\ (-1)^m \frac{2\pi i}{(m-1)!} n(n-1)\cdots(n-m+1), & \text{if } m-1 \le n. \end{cases} \end{aligned}$$

(c) Recall that $|dz| = i\rho\frac{dz}{z}$. First notice that if $a = 0$, then the given integral reduces to

$$-i\rho \int_C \frac{dz}{\rho^4 z} = \frac{2\pi}{\rho^3}.$$

Now we focus on the case when $a \neq 0$. Specifically, we we will consider the cases where $|a| > \rho$, and $|a| < \rho$. After much algebra, we obtain

$$\int_C \frac{|dz|}{|z-a|^4} = -\frac{i\rho}{\overline{a}^2}\int_C \frac{z\,dz}{(z-\frac{\rho}{\overline{a}})^2 (z-a)^2}.$$

Let $b = \frac{\rho}{\overline{a}}$. In the former case, we have that $z/(z-a)^2$ is analytic in C. Therefore,

$$\frac{2\pi\rho}{\overline{a}^2}\left(\frac{a+b}{(a-b)^3}\right).$$

The result turns out to be identical for $|a| < \rho$.

2. (Exercise 2, Section 4.2.3) Prove that a function that is analytic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some n and all sufficiently large $|z|$, reduces to a polynomial.

Proof. We can make the following estimate of the kth derivative of f at a point z:

$$\begin{aligned} |f^{(k)}(z)| &\le \frac{n!}{2\pi}\int_C \frac{|f(\zeta)|}{|\zeta - z|^{k+1}}\,|d\zeta| \\ &= n!\frac{|z|^n}{|z|^k} \\ &= n!|z|^{n-k}. \end{aligned}$$

Setting $k = n+1$, we have

$$|f^{(n+1)}(z)| \leq \frac{(n+1)!}{|z|} \longrightarrow 0$$

as $|z| \to \infty$. Since the above assertion holds for any z, we conclude that f must be a polynomial. $\square$

3. (Exercise 3, Section 4.2.3) If f is analytic and $|f(z)| \leq M$ for $|z| \leq R$, find an upper bound for $|f^{(n)}(z)|$ in $|z| \leq \rho < R$.

   Solution. Instead of using r in Cauchy's estimate, which corresponds to the radius of the circle centered at some point, we can now surround z (the point at which we evaluate the derivative $f^{(n)}$) by a circle of radius at most $R - \rho$. Therefore, we have the upper bound

$$|f^{(n)}(z)| \leq \frac{n!M}{(R-\rho)^n}$$

4. (Exercise 5, Section 4.2.3) Show that the successive derivatives of an analytic function at a point can never satisfy $|f^{(n)}(z)| > n!n^n$. Formulate a sharper theorem of the same kind.

   Proof. Let C be a circle of radius r centered at z such that f is analytic inside C (and on C). For the nth derivative of f, we know the estimate

$$\begin{aligned} |f^{(n)}(z)| &\leq \frac{n!}{2\pi} \int_C \frac{|f(\zeta)|}{|\zeta - z|^{n+1}}\, dz \\ &\leq \frac{n!M(2\pi r)}{(2\pi)r^{n+1}} \\ &\leq \frac{n!M}{r^n}, \end{aligned}$$

where we define M to be $\sup|f(\zeta)|$, which we know must be finite since we are taking the supremum of f over a compact set. Now let n be $\max\{1, M/r\}$, so we have $nr \geq M > 1$, which implies that $(nr)^n \geq M \Longrightarrow n^n \geq \frac{M}{r^n}$.

Substituting the above inequality into our estimate for the nth derivative of f yields $|f^{(n)}(z)| \leq n!n^n$, which implies that the derivative of f never satisfies the strict inequality $|f^{(n)}(z)| > n!n^n$.

$\square$