4.3.2 #(1,2,3,4) pg 130

1) if $f(z)$ and $g(z)$ have algebraic order h and k at $z=a$, show that fg has the order $h+k$, f/g order $h-k$ and $f+g$ an order which does not exceed $\max(h,k)$.

Pf: Taking the definitions on page 128, we know that the algebraic order of f at a is $\lim_{z \to a} |z-a|^{\alpha}|f(z)| = 0$ for $\alpha > h$ and

$$\lim_{z \to a} |z-\alpha|^{\alpha}|f(z)| = \infty \text{ for } \alpha < h.$$

Therefore we can see that if f has order h and g has order k at $z=a$ we have

$$|z-a|^{a}|f(z)g(z)| = \infty \quad \text{for } \alpha < h+k$$

and $|z-a|^{a}|f(z)g(z)| = 0$ for $\alpha > h+k$

Therefore the order of $f(z)g(z)$ is $h+k$.

Now, clearly if $g(z)$ has order k, then $\frac{1}{g(z)}$ has order $-k$. Therefore we know again $|z-a|^{\alpha}|f(z)/g(z)| = \infty$ for $\alpha < h-k$

and $|z-a|^{\alpha}|f(z)/g(z)| = 0$ for $\alpha > h-k$

→

Thus the order of $f(z)/g(z)$ is $h-k$.

Now we can see that, by the triangle inequality, we have

$$|z-a|^\alpha |f(z)+g(z)| \le |z-a|^\alpha |f(z)| + |z-a|^\alpha |g(z)| \xrightarrow[z \to a]{} 0$$

and as we know if f has order h and g has order k then

$f(z)+g(z)$ has order $\max(h,k)$. //

Show that a function which is analytic in the whole plane and has nonessential singularity at ∞ reduces to a polynomial.

Pf: Suppose that f has a removable singularity at ∞, then $f\left(\frac{1}{z}\right)$ has a removable singularity at $z=0$ and thus we know that $f\left(\frac{1}{z}\right)$ is bounded near 0. Therefore Given $|z|<\delta$, $\exists M, \delta>0$ such that $|f(1/z)| \leq M$ which implies that

$|f(z)| \leq M$ if $|z| > 1/\delta$.

Therefore f is an integral function and thus has to be bounded in the compact set $|z| \leq 1/\delta$. $\Rightarrow$ $|z| \leq L$ $\forall z \in \mathbb{C}$.

$\Rightarrow$ f is a bounded integral function

$\Rightarrow$ f is constant.

Furthermore, suppose f has a pole at ∞. Clearly, $f(1/z)$ has a pole at $z=0$. Thus we know $z^n f(1/z)$ is analytic at $z=0$, for some $n \in \mathbb{N}$. Now, using the fact that $f(z)$ has a pole at $z=a$ of order n if and only if $\rightarrow$

$n = \min \{ \alpha \in \mathbb{N} \mid |z-a|^{\alpha} f(z)$ is bounded in a neighborhood of $a \}$ then we have that $|f(1/z)| \leq a \left|\frac{1}{z}\right|^n$ for sufficiently small $|z|$ or $|f(z)| \leq a|z|^n$ for sufficiently large $|z|$.

Thus by the previous proof of problem 2 from page 123, we can see that $f(z)$ reduces to a polynomial. //

ow that functions e^z, $\sin z$, and $\cos z$ have
ential singularities at ∞.

Pf: Here, we can take the map e^z which will take a horizontal strip (infinite) with width $2\pi \xrightarrow{\text{to}} \mathbb{C} \setminus \{0\}$. therefore it takes any neighborhood $|z| > R$ of ∞ to $\mathbb{C} \setminus \{0\}$. well this implies that e^z does not have a limit (as $z \to \infty$) in $\hat{\mathbb{C}}$, therefore the isolated singularity is essential at ∞.

Similarly, for $\sin z$ and $\cos z$, we can take the line $z = 2k\pi + iy$ and we can see that $\cos z$ maps z to $[0, \infty)$ and $\sin z$ maps z to the Imaginary axis, therefore, as $R \to \infty$, the image of $|z| > R$ does not converge in $\hat{\mathbb{C}}$. //

#4| Show that any meromorphic function extended plane is rational. 5/5

Pf: Let $f(z)$ be any meromorphic function. We know that $f(z)$ can only have a finite # of poles in the extended plane. Otherwise, they would collect to a point which is not an isolated singularity.

Well, if $f(z)$ is analytic at ∞, then we can define $P(z)_\infty$ to be the constant function $f(\infty)$. (otherwise $f(z)$ has a pole at ∞ and $P(z)_\infty$ would be the principal of $f(z)$ at ∞).

We can see $P(z)_\infty$ is a polynomial and thus $(f(z) - P(z)_\infty) \longrightarrow 0$ as $z \to \infty$.

Call $\xi_1, \dots \xi_m$ the poles of $f(z)$ in the complex plane $\mathbb{C}$ and thus let $P_k(z)$ be the principal of $f(z)$ at ξ_k. Thus we write $P_k(z)$ as

$$P_k(z) = \frac{A_1}{z - \xi_k} + \frac{A_2}{(z - \xi_k)^2} + \dots + \frac{A_n}{(z - \xi_k)^n}$$

and note $P_k(z)$ is analytic at ∞. Thus consider the function $h(z) = f(z) - P(z)_\infty - \sum_{i=1}^{m} P_i(z)$

→

Hania Bykora

Since $f(z) - P_k(z)$ is analytic at ξ_k and each $P_i(z)$ is analytic at ξ_k for $i \neq k$ then $h(z)$ is analytic at ξ_k. Therefore $h(z)$ is an entire function and $\lim_{z \to \infty} h(z) = 0$. Thus by Liouville's Thm $\Rightarrow$ $h(z) = 0$. and we have

$$f(z) = P(z)_\infty + \sum_{i=1}^{m} P_i(z)$$ which is rational.

10/10

1 Find the residues at all singularites for the following fuctions:

(a) $\frac{z}{z^4+1}$

(b) $\frac{\sin z}{z^2(\pi-z)}$

(c) $\frac{ze^{iz}}{(z-\pi)^2}$

(d) $\frac{z^3+5}{(z^4-1)(z+1)}$

(a) There are four poles, the fourth roots of -1, which are $\frac{1+i}{\sqrt{2}}$, $\frac{1-i}{\sqrt{2}}$, $\frac{-1+i}{\sqrt{2}}$ and $\frac{-1-i}{\sqrt{2}}$. Each has order 1. For $z_0 = \frac{1+i}{\sqrt{2}}$, we calculate

$$\begin{aligned}
\operatorname{Res}_{z=\frac{1+i}{\sqrt{2}}} \frac{z}{z^4+1} &= \lim_{z\to\frac{1+i}{\sqrt{2}}} \left(z - \frac{1+i}{\sqrt{2}}\right) \frac{z}{z^4+1} \\
&= \lim_{z\to\frac{1+i}{\sqrt{2}}} \frac{z}{\left(z-\frac{1-i}{\sqrt{2}}\right)\left(z-\frac{-1+i}{\sqrt{2}}\right)\left(z-\frac{-1-i}{\sqrt{2}}\right)} \\
&= \frac{\frac{1+i}{\sqrt{2}}}{\left(\frac{1+i}{\sqrt{2}}-\frac{1-i}{\sqrt{2}}\right)\left(\frac{1+i}{\sqrt{2}}-\frac{-1+i}{\sqrt{2}}\right)\left(\frac{1+i}{\sqrt{2}}-\frac{-1-i}{\sqrt{2}}\right)} \\
&= \frac{2(1-i)}{(2i)\,2\,(2i+2)} = \frac{1}{4i} = \frac{-i}{4}.
\end{aligned}$$

Similarly, for $z_0 = \frac{1-i}{\sqrt{2}}$ we calculate

$$\begin{aligned}
\operatorname{Res}_{z=\frac{1-i}{\sqrt{2}}} \frac{z}{z^4+1} &= \lim_{z\to\frac{1-i}{\sqrt{2}}} \left(z - \frac{1-i}{\sqrt{2}}\right) \frac{z}{z^4+1} \\
&= \lim_{z\to\frac{1-i}{\sqrt{2}}} \frac{z}{\left(z-\frac{1+i}{\sqrt{2}}\right)\left(z-\frac{-1+i}{\sqrt{2}}\right)\left(z-\frac{-1-i}{\sqrt{2}}\right)} \\
&= \frac{\frac{1-i}{\sqrt{2}}}{\left(\frac{1-i}{\sqrt{2}}-\frac{1+i}{\sqrt{2}}\right)\left(\frac{1-i}{\sqrt{2}}-\frac{-1+i}{\sqrt{2}}\right)\left(\frac{1-i}{\sqrt{2}}-\frac{-1-i}{\sqrt{2}}\right)} \\
&= \frac{2(1-i)}{(-2i)\,2\,(-2i+2)} = \frac{1}{-4i} = \frac{i}{4}.
\end{aligned}$$

Similarly, for $z_0 = \frac{-1+i}{\sqrt{2}}$ we calculate

$$\begin{aligned}
\operatorname{Res}_{z=\frac{-1+i}{\sqrt{2}}} \frac{z}{z^4+1} &= \lim_{z\to\frac{-1+i}{\sqrt{2}}} \left(z - \frac{-1+i}{\sqrt{2}}\right)\frac{z}{z^4+1} \\
&= \lim_{z\to\frac{-1+i}{\sqrt{2}}} \frac{z}{\left(z-\frac{1+i}{\sqrt{2}}\right)\left(z-\frac{1-i}{\sqrt{2}}\right)\left(z-\frac{-1-i}{\sqrt{2}}\right)} \\
&= \frac{\frac{-1+i}{\sqrt{2}}}{\left(\frac{-1+i}{\sqrt{2}}-\frac{1+i}{\sqrt{2}}\right)\left(\frac{-1+i}{\sqrt{2}}-\frac{1-i}{\sqrt{2}}\right)\left(\frac{-1+i}{\sqrt{2}}-\frac{-1-i}{\sqrt{2}}\right)} \\
&= \frac{2(-1+i)}{(2i)(-2)(2i-2)} = \frac{1}{-4i} = \frac{i}{4}.
\end{aligned}$$

Finally, for $z_0 = \frac{-1-i}{\sqrt{2}}$ we calculate

$$\begin{aligned}
\operatorname{Res}_{z=\frac{-1-i}{\sqrt{2}}} \frac{z}{z^4+1} &= \lim_{z\to\frac{-1-i}{\sqrt{2}}} \left(z - \frac{-1-i}{\sqrt{2}}\right)\frac{z}{z^4+1} \\
&= \lim_{z\to\frac{-1-i}{\sqrt{2}}} \frac{z}{\left(z-\frac{1+i}{\sqrt{2}}\right)\left(z-\frac{1-i}{\sqrt{2}}\right)\left(z-\frac{-1+i}{\sqrt{2}}\right)} \\
&= \frac{\frac{-1-i}{\sqrt{2}}}{\left(\frac{-1-i}{\sqrt{2}}-\frac{1+i}{\sqrt{2}}\right)\left(\frac{-1-i}{\sqrt{2}}-\frac{1-i}{\sqrt{2}}\right)\left(\frac{-1-i}{\sqrt{2}}-\frac{-1+i}{\sqrt{2}}\right)} \\
&= \frac{2(-1-i)}{(-2i)(-2)(-2i-2)} = \frac{1}{4i} = \frac{-i}{4}.
\end{aligned}$$

(b) There are poles at 0 and π. The pole at $z_0 = 0$ has order 2, and we calculate

$$\begin{aligned}
\operatorname{Res}_{z=0} \frac{\sin z}{z^2(\pi-z)} &= \frac{d}{dz} z^2 \frac{\sin z}{z^2(\pi-z)}\bigg|_{z=0} \\
&= \frac{d}{dz}\frac{\sin z}{(\pi-z)}\bigg|_{z=0} \\
&= \frac{\cos(z)(\pi-z)-\sin(z)(-1)}{(\pi-z)^2}\bigg|_{z=0} \\
&= \frac{\cos(0)(\pi)+\sin(0)}{\pi^2} = \frac{\pi}{\pi^2} = \frac{1}{\pi}.
\end{aligned}$$

The pole at $z_0 = \pi$ has order 1 and we calculate

$$\begin{aligned}
\operatorname{Res}_{z=\pi} \frac{\sin z}{z^2(\pi-z)} &= \lim_{z\to\pi}(z-\pi)\frac{\sin z}{z^2(\pi-z)} \\
&= \lim_{z\to\pi}\frac{-\sin z}{z^2} \\
&= \frac{-\sin(\pi)}{\pi^2} = \frac{0}{\pi^2} = 0.
\end{aligned}$$

(c) The only pole is $z_0 = \pi$ with order 2. Then we calculate

$$\begin{aligned}\mathrm{Res}_{z=\pi}\frac{ze^{iz}}{(z-\pi)^2} &= \frac{d}{dz}(z-\pi)^2\frac{ze^{iz}}{(z-\pi)^2}\bigg|_{z=\pi}\\ &= \frac{d}{dz}ze^{iz}\bigg|_{z=\pi}\\ &= e^{iz}+ize^{iz}\big|_{z=\pi} = e^{i\pi}+i\pi e^{i\pi} = -1-i\pi.\end{aligned}$$

(d) The poles are at 1, -1, i and $-i$. We start with $z_0 = -1$, which has order 2. Then the residue is given by

$$\begin{aligned}\mathrm{Res}_{z=-1}\frac{z^3+5}{(z^4-1)(z+1)} &= \frac{d}{dz}(z+1)^2\frac{z^3+5}{(z^3-z^2+z-1)(z+1)^2}\bigg|_{z=-1}\\ &= \frac{d}{dz}\frac{z^3+5}{(z^3-z^2+z-1)}\bigg|_{z=-1}\\ &= \frac{3z(z^3-z^2+z-1)-(3z^2-2z+1)(z^3+5)}{(z^3-z^2+z-1)^2}\bigg|_{z=-1}\\ &= \frac{(-3)(-4)-(6)(4)}{(-4)^2} = \frac{-36}{16} = \frac{-9}{4}.\end{aligned}$$

The other poles all have order 1, and we calculate for $z_0 = 1$

$$\begin{aligned}\mathrm{Res}_{z=1}\frac{z^3+5}{(z^4-1)(z+1)} &= \lim_{z\to1}(z-1)\frac{z^3+5}{(z^4-1)(z+1)}\\ &= \lim_{z\to1}\frac{z^3+5}{(z+1)^2(z-i)(z+i)}\\ &= \frac{1+5}{4(1-i)(1+i)} = \frac{6}{2(1+1)} = \frac{3}{4}.\end{aligned}$$

At $z_0 = i$, we have

$$\begin{aligned}\mathrm{Res}_{z=i}\frac{z^3+5}{(z^4-1)(z+1)} &= \lim_{z\to i}(z-i)\frac{z^3+5}{(z^4-1)(z+1)}\\ &= \lim_{z\to i}\frac{z^3+5}{(z+1)^2(z-1)(z+i)}\\ &= \frac{-i+5}{(-1+2i+1)(-1-1-i-i)}\\ &= \frac{-i+5}{(2i)(-2-2i)} = \frac{-i+5}{-4i+4} = \frac{3}{4}+\frac{i}{2}.\end{aligned}$$

Finally, at $z_0 = -i$, we have

$$\begin{aligned}
\operatorname{Res}_{z=-i} \frac{z^3+5}{(z^4-1)(z+1)} &= \lim_{z\to -i} (z+i) \frac{z^3+5}{(z^4-1)(z+1)} \\
&= \lim_{z\to -i} \frac{z^3+5}{(z+1)^2 (z-1)(z-i)} \\
&= \frac{i+5}{(-1-2i+1)(-1-1-i-i)} \\
&= \frac{-i+5}{(2i)(-2+2i)} = \frac{i+5}{4i+4} = \frac{3}{4} - \frac{i}{2}.
\end{aligned}$$

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2 Let C denote the unit circle, traversed counter-clockwise. Compute each of the following integrals:

$$\int_C \frac{e^{\pi z}}{4z^2+1} dz, \qquad \int_C \frac{e^z}{\left(z^2+z-\frac{3}{4}\right)^2} dz$$

For the first integral, we have poles at $z_0 = \pm\frac{i}{2}$. Then we calculate the residue of $z_0 = \frac{i}{2}$, which has order 1:

$$\begin{aligned}
\operatorname{Res}_{z=\frac{i}{2}} \frac{e^{\pi z}}{4z^2+1} &= \lim_{z\to \frac{i}{2}} \left(z - \frac{i}{2}\right) \frac{e^{\pi z}}{4z^2+1} \\
&= \lim_{z\to \frac{i}{2}} \left(z - \frac{i}{2}\right) \frac{e^{\pi z}}{(2z+i)(2z-i)} \\
&= \frac{1}{2} \lim_{z\to \frac{i}{2}} (2z-i) \frac{e^{\pi z}}{(2z+i)(2z-i)} \\
&= \frac{1}{2} \lim_{z\to \frac{i}{2}} \frac{e^{\pi z}}{2z+i} = \frac{e^{\frac{i\pi}{2}}}{2(i+i)} = \frac{i}{4i} = \frac{1}{4}.
\end{aligned}$$

Similarly, for $z_0 = -\frac{i}{2}$, we have

$$\begin{aligned}
\operatorname{Res}_{z=-\frac{i}{2}} \frac{e^{\pi z}}{4z^2+1} &= \lim_{z\to -\frac{i}{2}} \left(z + \frac{i}{2}\right) \frac{e^{\pi z}}{4z^2+1} \\
&= \lim_{z\to -\frac{i}{2}} \left(z + \frac{i}{2}\right) \frac{e^{\pi z}}{(2z+i)(2z-i)} \\
&= \frac{1}{2} \lim_{z\to -\frac{i}{2}} (2z+i) \frac{e^{\pi z}}{(2z+i)(2z-i)} \\
&= \frac{1}{2} \lim_{z\to -\frac{i}{2}} \frac{e^{\pi z}}{2z-i} = \frac{e^{\frac{-i\pi}{2}}}{2(-i-i)} = \frac{-i}{-4i} = \frac{1}{4}.
\end{aligned}$$

Then by the Residue Theorem, we have

$$\int_C \frac{e^{\pi z}}{4z^2+1} dz = 2\pi i \left(\frac{1}{4} + \frac{1}{4}\right) = \pi i.$$

For the second integral, there are poles at $z_0 = \frac{1}{2}$ and $z_0 = -\frac{3}{2}$. Since $n\left(C; -\frac{3}{2}\right) = 0$, we only need to calculate the residue at $z_0 = \frac{1}{2}$. The poles $z_0 = \frac{1}{2}$ has order 2, so we calculate

$$
\begin{aligned}
\mathrm{Res}_{z=\frac{1}{2}} \frac{e^z}{\left(z^2+z-\frac{3}{4}\right)^2} &= \frac{d}{dz}\left(z-\frac{1}{2}\right)^2 \frac{e^z}{\left(z^2+z-\frac{3}{4}\right)^2}\Bigg|_{z=\frac{1}{2}} \\
&= \frac{d}{dz}\frac{e^z}{\left(z+\frac{3}{2}\right)^2}\Bigg|_{z=\frac{1}{2}} \\
&= \frac{d}{dz}\frac{4e^z}{(2z+3)^2}\Bigg|_{z=\frac{1}{2}} \\
&= 4\,\frac{e^z(2z+3)^2 - 4(2z+3)e^z}{(2z+3)^4}\Bigg|_{z=\frac{1}{2}} \\
&= 4\,\frac{e^z(2z+3) - 4e^z}{(2z+3)^3}\Bigg|_{z=\frac{1}{2}} \\
&= 4e^z\,\frac{2z-1}{(2z+3)^3}\Bigg|_{z=\frac{1}{2}} \\
&= 4\sqrt{e}\frac{0}{(1+3)^3} = 0.
\end{aligned}
$$

Therefore, by the Residue Theorem,

$$
\int_C \frac{e^z}{\left(z^2+z-\frac{3}{4}\right)^2}dz = 0.
$$

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3 Riemann integrals of a real variable θ of the form

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

can sometimes be solved by changing variables to a complex contour integral. Indeed, setting $z = e^{i\theta}$, then as θ runs from 0 to 2π, z traverses the unit circle counterclockwise. Moreover,

$$\cos\theta = \frac{z^2+1}{2z}, \quad \sin\theta = \frac{z^2-1}{2iz}, \quad d\theta = \frac{dz}{iz}.$$

Show that such a change of variables allows one to evaluate:

$$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$$

via a complex countour integral, upon using the residue theorem.

We make the substitutions suggested to get

$$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \int_C \frac{dz}{iz\left(2+\frac{z^2-1}{2iz}\right)} = \int_C \frac{dz}{iz\left(\frac{4iz+z^2-1}{2iz}\right)} = \int_C \frac{2dz}{z^2+4iz-1}.$$

Then the poles z_0 of the integrand can be found using the quadratic formula:

$$z_0 = \frac{-4i \pm \sqrt{-16+4}}{2} = -2i \pm i\sqrt{3} = i\left(-2 \pm \sqrt{3}\right).$$

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Since $\left|i\left(-2-\sqrt{3}\right)\right| > 1$, we have $n\left(C; i\left(-2-\sqrt{3}\right)\right) = 0$, and we only need to compute the residue of $z_0 = i\left(-2+\sqrt{3}\right)$. The order of the pole is 1, so we compute

$$\begin{aligned}
\mathrm{Res}_{z=i(-2+\sqrt{3})} \frac{2}{z^2+4iz-1} &= \lim_{z \to i(-2+\sqrt{3})} \left(z - i\left(-2+\sqrt{3}\right)\right) \frac{2}{z^2+4iz-1} \\
&= \lim_{z \to i(-2+\sqrt{3})} \frac{2}{z - i\left(-2-\sqrt{3}\right)} \\
&= \frac{2}{i\left(-2+\sqrt{3}\right) + i\left(2+\sqrt{3}\right)} = \frac{2}{2i\sqrt{3}} = \frac{-i}{\sqrt{3}}.
\end{aligned}$$

Therefore, by the Residue Theorem,

$$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = 2\pi i \frac{-i}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

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