Plan for the day: - a few more consequences of C.I.F.

- discuss what's next after midterm
- reminders about content of midterm.

Consequences of C.I.F. so far:

- Morera's thm (ints. of f vanish in $\Omega \Rightarrow f$ analytic)

- Liouville's thm (bounded, entire functions constant)

- Cauchy's estimate on n^{th} derivatives

swap order

- Analytic functions have derivs of all orders

- (finite) Taylor approxs: f analytic on $\Omega \ni a$

notation not so great, since f_n depends on a.

$$f(z) = f(a) + f'(a)(z-a) + \cdots + \frac{f^{n-1}(a)(z-a)^{n-1}}{(n-1)!} + (z-a)^n \cdot \underbrace{f_n(z)}_{\text{holom. function on } \Omega}$$

with $$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{(\xi-a)^n(\xi-z)}\, d\xi$$

Perform Cauchy-ineq.-type estimate on this. f continuous on C (compact) means $\underbrace{|f(z)|}_{\substack{\text{continuous, real-valued} \\ \text{(maps compacts to compacts of } \mathbb{R})}} \leq M$

then $$|f_n(z)| \leq \frac{M}{R^{n-1}\cdot(R-|z-a|)}$$

if C is centered at a of radius R.

and z inside C so $|z-a| < R$.

Thm: f analytic in Ω. $a \in \Omega$ s.t. $f^{(n)}(a) = 0 \ \forall\, n \geq 0$

then $f \equiv 0$ in Ω.

identically 0 at all pts in Ω.

pf: Since derivatives at a vanish,

$$f(z) = (z-a)^n f_n(z) \quad \text{for any } n.$$

By our estimate

$$|f(z)| \leq \frac{|z-a|^n}{R^n} \cdot \frac{MR}{(R-|z-a|)}.$$

Take limit as $n \to \infty$

$\frac{|z-a|}{R} < 1$ so

$\Rightarrow f(z) = 0$ on interior of C.

Left to show: f is 0 on all of Ω.

(C was just circle centered about a inside Ω)

Clever topological arg.: $\Omega = E_1 \cup E_2$

E_1: pts. $z_0 \in \Omega$ s.t. all derivs of f vanish ~~all derivs of f at $z_0 \in E_1$ vanish.~~

E_2: pts. z_0 s.t. some deriv. doesn't vanish at z_0

previous argument via circles $\Rightarrow$ E_1 open.

E_2 open since derivs of f are continuous, so $f^{(k)}(z_0) \neq 0$

say equal to z_1

take open nbhd of z_1 not containing 0. inv. image under $f^{(k)}$ is open.

But Ω open, conn. $\Rightarrow$ either E_1, E_2 empty.

But $a \in E_1$ so

must be E_2 is empty.

$\Rightarrow f \equiv 0$ in Ω.

Turn this logic around:

If $f \not\equiv 0$ on Ω, then smallest h s.t. $f^{(h)}(a) \neq 0$ at any $a \in \Omega$.

(i.e. order of a zero of f is finite, for all analytic functions)

Write it $f(z) = (z-a)^h f_h(z)$

with $f_h(a) \neq 0$ just as for polynomials.

In fact, $f_h(z) \neq 0$ in nbhd of a

(since $f_h(z)$ analytic, so also continuous)

so zeros of (non-zero) analytic function are isolated. [ALL or NOTHING rule]

Corollary If f, g analytic on Ω and $f(z) = g(z)$ for $z \in S$: set with accumulation point $\in \Omega$

then $f = g$ for all $z \in \Omega$.

pf: For $z \in S$, $f - g = 0$ so zeros of $f-g$ not isolated

$\Rightarrow f - g \equiv 0$ on Ω.