~~now use characterization of sing.: $\nexists\ \beta$ s.t. $\lim_{z \to a} |z-a|^{\beta} |f(z) - w_0| = 0$~~

~~$\Rightarrow \lim_{z \to a} |z-a|^{\beta} |f(z)| = 0$ so f does not have ess. sing. at a. $\lightning$.~~

Other uses of isolated zeros / Cauchy int. formula:

Suppose f analytic on disk D, γ: closed curve in D with
(non-zero)
with finitely many zeros $z_1, \dots, z_n$ $\quad z_i \notin \gamma$.

Write $f(z) = (z - z_1) \cdots (z - z_n)\, g(z)$ with $g(z)$ analytic, $\neq 0$ on D.

(Here z_i's may not be distinct.
can be repeated according to multiplicity)

Just as in Lucas-Gauss thm. on zero's of polynomials:

Take logarithmic derivative to get

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \cdots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)}.$$

Integrate both sides over γ.

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)}\, dz = \sum_j \underbrace{n(\gamma, z_j)}_{\text{winding number}}.$$

(since $\int_{\gamma} \frac{g'(z)}{g(z)} = 0$
b/c $g(z) \neq 0$ on D
so $g'(z)/g(z)$ is analytic on D)

Here use that we only know Cauchy's thm on disk D.

Note that our assumption about f having finitely many zeros is unnecessary since only zeros inside γ contribute to equality. There are only finitely many of these since $\gamma \subset D' \subset D$ for some D' with $\bar{D}' \subset D$. On this compact set $\bar{D}'$, infinitely many zeros would have an accumulation point.

If curve is simple, e.g. circle, then $n(\gamma, z_j)$ will always be $0, 1$

so can think of $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$ as # of zeros (w/ mult.) inside γ.

Slight generalization: Apply to $f(z) - a$:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-a}\,dz = \sum_j n(\gamma, z_j(a))$$

$z_j(a)$ are pts. for which $f(z_j(a)) = a$.
i.e. sol'ns to $f(z) = a$.

(need that $f(z) \neq a$ on γ, of course)

Slight reformulation: $f : \gamma \to \sigma$: closed curve, write $w = f(z)$ then think of σ as lying in w-plane.

$$n(\sigma; a) \overset{\text{def}}{=} \frac{1}{2\pi i}\int_\sigma \frac{dw}{w-a}$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)-a} = \sum_j n(\gamma, z_j(a))$$

Since $n(\sigma; a)$ is constant on regions defined by σ, then

if a, b in same region: $\sum_j n(\gamma, z_j(a)) = \sum_j n(\gamma, z_j(b))$

In words, f takes values a, b same number of times inside γ, if a suff. close to b.

Thm: $f(z)$ analytic in nbhd of z_0, $f(z_0) = w_0$, with $f(z_0) - w_0$ having 0 of order n at z_0.
For suff. small $\epsilon > 0$, $\exists\ \delta > 0$ s.t. for all a with $|a - w_0| < \delta$
$f(z) = a$ has n roots in a disk $|z - z_0| < \epsilon$.

in part. $f(z) \neq w_0$
i.e. non-constant

③ We know that if a, b close in w-plane, then $\sum_j n(\gamma, z_j(a)) = \sum_i n(\gamma, z_i(b))$ (*)

As a special case for use in theorem, take γ = circle of radius ε around single solution z_0 to $f(z) = w_0$.

$B(z_0, \varepsilon)$.

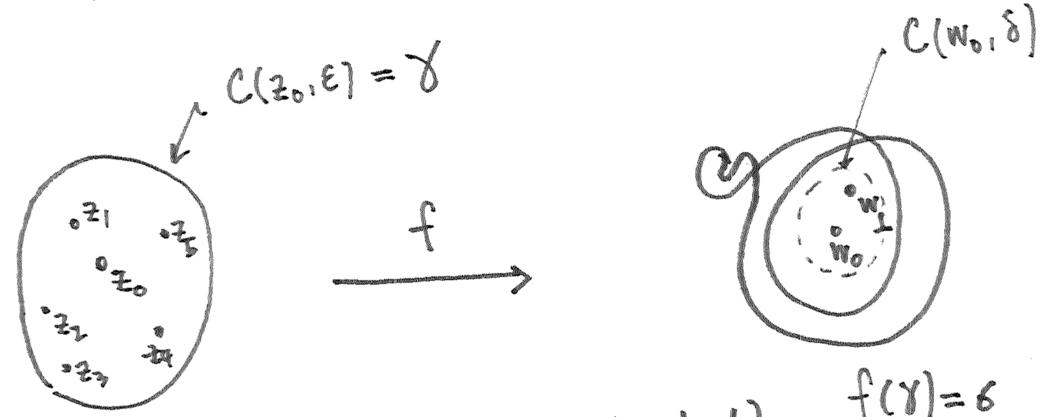
Want to show $f(B(z_0,\varepsilon)) \supseteq B(w_0,\delta)$ for δ suff. small.

Pick point $w_1 \in B(w_0,\delta)$. Want to show $\exists\ z_1 \in B(z_0,\varepsilon)$ s.t. $w_1 = f(z_1)$.

But (*) proves more precisely that if z_0 is solution of order n to $f(z) - w_0$, then there are n points in $B(z_0,\varepsilon)$ for which $f(z_i) = w_1$.
$z_1, \dots, z_n$

Picture:

(with, say, z_0 a sol'n to $f(z) - w_0$ of order 5)



$z_1, \dots, z_5 \longmapsto w_1$ (sol'ns of order 1)

$z_0 \longmapsto w_0$ (sol'n of order 5) pf: choose ϵ so that z_0 is only 0 of ~~[illegible]~~ $f(z)-w_0$ in this nbhd.

(can be done since zeros isolated). Apply above result with $\gamma = C(z_0, \epsilon)$
circle centered at z_0 radius ϵ.

Let $\sigma = f(\gamma)$. By construction $f(z) \neq w_0$ on σ so the result applies.

(Indeed $f(z)$ analytic, so has isolated zeros.)

So $\exists$ $\delta > 0$ s.t. $B(w_0, \delta) \cap \sigma = \phi$. This is desired δ.

(Note: by picking ϵ suff. small, can assume sol'ns to $f(z) = a$ have multiplicity 1.

Corollary: (Open Mapping Thm) Ω: open, conn., f: non-const. analytic function on Ω

then f maps open sets in Ω to open sets.

(since previous thm showed $f(B(z_0, \epsilon)) \supset B(w_0, \delta)$)

Corollary 2: (Maximum principle) f: analytic, non-const. on Ω, then

$|f(z)|$ has no maximum on Ω.

pf: Given $w_0 = f(z_0)$, then $\exists$ $B(w_0, \delta) \subseteq f(B(z_0, \epsilon))$

So $\exists$ point $w \in B(w_0, \delta)$ with $|w| > |w_0|$. So $f(z_0)$ not maximum. //

Alternate (positive) formulation: $f(z)$: continuous on compact set E, analytic on interior, then $\max |f(z)|$ is attained on the boundary of E.

pf: $|f|$ has a max on E, since E assumed compact. If max occurs at z_0 in interior, then f must be constant on component of E containing z_0.

(so max also attained on boundary of this component, $\subseteq$ boundary of E)