Maximum modulus principle : f defined, continuous on E: compact and analytic on interior of E, then $\max_{z\in E} |f(z)|$ is attained on the boundary.

(so if $|f(z)| \leq M$ on ∂E, then $|f(z)| \leq M$ on E)

More definitive version : Ω: region in $\mathbb{C}$, f analytic on Ω, s.t. $\limsup_{z\to a} |f(z)| \leq M$ $\forall\, a \in \partial_\infty \Omega$, then $|f(z)| \leq M$ on Ω (i.e. $\forall\, z\in\Omega$)

(Here $\limsup_{z\to a} |f(z)| = \lim_{r\to 0^+} \sup \{ f(z) \mid z\in \Omega\cap B(a,r) \}$)

Also $\partial_\infty \Omega = \begin{cases} \partial\Omega & \text{if } \Omega \text{ bounded} \\ \partial\Omega \cup \{\infty\} & \text{if } \Omega \text{ unbounded.} \end{cases}$

pf: Given any $\delta>0$, $\Phi := \{ z\in\Omega \mid |f(z)| > M+\delta \}$. Want to show Φ empty. Φ open since $|f(z)|$ is continuous, and $\overline{\Phi} \subseteq \Omega$ since, for each $a\in\partial_\infty(\Omega)$, $\limsup_{z\to a} |f(z)| \leq M$ so $\exists$ $B(a;\varepsilon)$ s.t. $|f(z)| < M+\delta$ $\forall\, z\in\Omega\cap B(a,\varepsilon)$

Finally, Φ bounded by applying this condition with $a=\infty$.
(nbhds of ∞ have complements : bounded)

So Φ compact. Apply maximum modulus principle.

If $z \in \partial(\Phi)$, then $|f(z)| = M + \delta$ since f continuous.

$\Phi = \phi$. //

and Φ defined by condition that $|f(z)| > M+\delta$.

Applying maximum modulus principle, either $f(z)$ constant (i.e. $f(z) = c$ with $|c| = M+\delta$)

or $f(z) \leq M+\delta$. $\Rightarrow$ (either way) $\Phi = \phi$.