Meromorphic functions and their complex line integrals.

f: analytic in a neighborhood of z_0, except possibly at z_0 itself.

i.e. in $0 < |z - z_0| < \delta$.

Examine $\lim_{z \to z_0} |(z - z_0)^k f(z)|$ for various k.

Earlier theorem: f has removable singularity at z_0 (define $f(z_0) = \lim_{z \to z_0} f(z)$)

$$\Longleftrightarrow \quad \lim_{z \to z_0} |(z - z_0)^1 f(z)| = 0.$$

this means defined and analytic on $\Omega \setminus \{z_0\}$ initially. Want to extend to all of Ω.

(pf. define $f(z_0) = \frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(\xi)}{\xi - z_0} \, d\xi$)

then treat $f(z)$ like analytic function once we've "plugged the hole"

in particular, this condition will be satisfied if $\lim_{z \to z_0} |f(z)| < \infty$.

also necessary since C.I.F $\Rightarrow f(z_0) \le \sup_C |f|$

[Compare real case: $|x|$ is differentiable on $\mathbb{R} - \{0\}$, but has no extension to diff. function on $\mathbb{R}$.]

Otherwise, two possibilities: $\underbrace{\lim_{z \to a} f(z) = \infty}_{\text{"pole"}}$, $\underbrace{\lim_{z \to a} f(z) \text{ does not exist}}_{\text{"essential singularity" wild behavior.}}$

Investigate poles: if $\lim_{z \to a} f(z) = \infty$, then for any $M > 0$,

find δ s.t. $|f(z)| \geq M$ on $|z - z_0| < \delta$ (and $0<$), in particular

$f(z) \neq 0$ on $B(z_0, \delta)$. Consider $g(z) = \dfrac{1}{f(z)}$ on this nbhd,

which is analytic by our analysis.

Since $\lim_{z \to z_0} g(z) = 0$, then so is $\lim_{z \to z_0} (z - z_0) g(z)$ and hence the singularity is removable, setting $g(z_0) = 0$.

Now $g(z)$ analytic but not $\equiv 0$ on $B(z_0, \delta)$

so expressible as $g(z) = (z - z_0)^h \underbrace{g_h(z)}$

analytic, $g_h(z_0) \neq 0$.

Remembering $g(z) = 1/f(z)$, then $f(z) = (z - z_0)^{-h} \underbrace{f_h(z)}$

analytic since $= 1/g_h(z)$ and $g_h(z_0) \neq 0$.

Functions that are analytic except for possibly isolated poles are called "meromorphic functions"

This discussion shows they behave like rational functions in same way that polynomials $\longleftrightarrow$ analytic functions.

f, g analytic, then f/g meromorphic with poles at (potentially) 0's of g (though they may cancel with 0's of f)

e.g. $f = \sin z$, $g = z$.

potential pole at $z = 0$ in $\sin z / z$ cancelled by 0 of $\sin z$.

Interesting consequence of our discussions: $\exists$ integer h s.t.

$$(*) \quad \lim_{z \to z_0} \left| (z - z_0)^{\alpha} f(z) \right| = 0 \quad \forall \text{ real } \alpha > h.$$

(for any pole of f)

$$(**) \quad \text{and} \quad \lim_{z \to z_0} \left| (z - z_0)^{\alpha} f(z) \right| = \infty \quad \forall \text{ real } \alpha < h.$$

If $f(z) = (z - z_0)^{-h} f_h(z)$, do Taylor expansion for

$f_h(z) = a_{-h} + a_{-(h-1)} (z - z_0) + \cdots$ so $f(z)$ expressible as:

$$f(z) = \underbrace{a_{-h} (z - z_0)^{-h} + \cdots + a_{-1} (z - z_0)^{-1}}_{\text{"singular part of } f\text{" @ } z_0} + \underbrace{\phi(z)}_{\text{analytic in nbhd of } z_0}$$

(If $f(z) = g(z)/h(z)$ g, h polys, then performing this expansion at each successive 0 of h is just the partial fractions expansion)

Essential singularities: (Neither $(*)$ nor $(**)$ hold for any α)

$\Rightarrow$ f unbounded in nbhd of z_0, an essential singularity (failure for pos α) and yet comes arbitrarily close to 0 (by failure of $(*)$ for any α)

Thm: An analytic function comes arbitrarily close to any complex value in a neighborhood of an essential singularity.

Last week: $f(z)$ analytic in Ω: open, conn., then $\int_\gamma f\, dz = 0$

for any cycle $\gamma \sim 0$. in Ω

Remember: cycle is just formal linear combination of closed paths (modulo equivalence)

Cor: (Stronger form of Cauchy Integral Formula) f, Ω as above.

$$n(\gamma, a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)\, dz}{z-a} \quad \text{if } \gamma \sim 0 \text{ in } \Omega.$$

(remember Cauchy Int. Formula is direct consequence of Cauchy's Thm for

$\phi(z) = \dfrac{f(z) - f(a)}{z-a}$, which has a removable singularity at $z = a$, but this causes no difficulty.)

Now suppose f: analytic on Ω except for finitely many isolated singularities $a_1, \dots, a_n$. So region we're working with is $\Omega \setminus \{a_1, \dots, a_n\}$

Let P_j be period corresponding to a_j.

That is $P_j(f) = \int_{C(a_j, \delta)} f(z) dz$ $\quad C(a_j, \delta)$: circle centered at a_j with δ suff. small.

e.g. $f(z) = \dfrac{1}{z - a_j}$. Then $\displaystyle\int_{C(a_j, \delta)} \frac{1}{z-a_j} dz = 2\pi i.$

For general function f, then

$$f(z) - \frac{P_j(f)}{2\pi i (z - a_j)}$$

constant indep. of z

has zero period around a_j

If we set $R_j(f) = \dfrac{P_j(f)}{2\pi i}$ then result, called Residue of f at a_j,

is unique cx. # such that $f(z) - \dfrac{R_j}{z-a_j}$ is derivative of (single-valued) analytic function on suffic. small nbhd of a_j.

Given any $\gamma \subseteq \Omega$ with $\gamma \sim 0$, then in the set $\Omega \setminus \{a_1, \dots, a_n\}$

$$\gamma \sim \sum_j n(\gamma, a_j) \cdot C_j \qquad C_j := C(a_j, \delta) \text{ : cycle with winding \# 1 around } a_j.$$

where $n(\gamma, a_j)$ is the winding # in $\Omega \setminus \{a_1, \dots, a_n\}$.

So then in this region:
$$\int_\gamma f\,dz = \int_{\sum_j n(\gamma,a_j) C_j} f\,dz = \sum_j n(\gamma, a_j)\, P_j$$

or, using residues $R_j = P_j / 2\pi i$:
$$\frac{1}{2\pi i} \int_\gamma f\,dz = \sum_j n(\gamma, a_j)\, R_j$$

Known as "Residue Thm". Note it applies equally well if f has infinitely many isolated zeros. Indeed, we just need to show that for any γ, $n(\gamma, a_j) = 0$ for all but fin. many j. This is so because $n(\gamma, a) = 0$ for any a in unbounded component of a. (i.e. other components bounded, say inside disk of radius R, for $R \gg 0$.) But then isolated singularities in this disk must be finite - else they would have an accumulation pt.