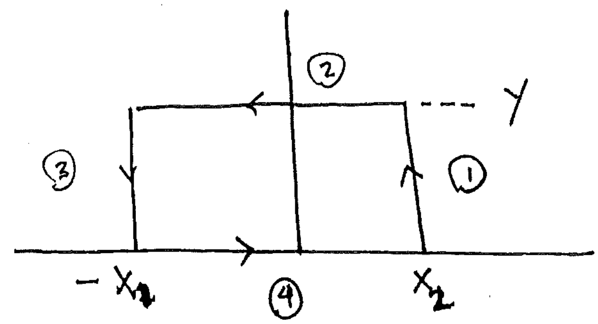
As in example 3, consider $\displaystyle\int_{-\infty}^{\infty} e^{ix}\,\frac{g(x)}{h(x)}\,dx$ with $\deg(h)-\deg(g)=1$.

choose a slightly different contour:



By degree condition,

$$\left| z\,\frac{g(z)}{h(z)} \right| \text{ is bounded } \quad \left(\text{i.e. } \left|\frac{g(z)}{h(z)}\right| \le M\cdot\frac{1}{|z|} \right)$$

some constant M

X_1, X_2, Y suff. large.

so
$$\left|\int_{①} f(z)\,dz\right| \le M'\int_0^Y e^{-y}\,\frac{dy}{|z|}$$

$$< M'\,\frac{1}{X_2}\int_0^Y e^{-y}\,dy < \frac{M'}{X_2}$$

similarly, $\displaystyle\left|\int_{③} f(z)\,dz\right| < \frac{M''}{X_1}$

$$\left|\int_{②} f(z)\,dz\right| \le \int_{②} e^{-Y}\,\frac{M'''}{|z|}\,|dz| < e^{-Y}\cdot M'''\,\frac{(X_1+X_2)}{Y}$$

since $|z|>Y$ on ②.

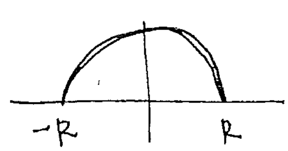
If we take $Y\to\infty$, with X_1, X_2 fixed, then $\displaystyle\left|\int_{②} f(z)\,dz\right| \to 0$.

What remains is $\displaystyle\left|\int_{-X_1}^{X_2} e^{ix}\,\frac{g(x)}{h(x)}\,dx - 2\pi i\sum \operatorname{Res}(f(z))\right| < M''''\left(\frac{1}{X_1}+\frac{1}{X_2}\right)$

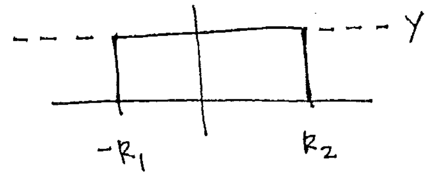
So far: $f = g/h$ with $\deg(h) - \deg(g) \geq 2$

or $f = e^{iz} g/h$ with $\deg(h) - \deg(g) \geq 1$ $\qquad e^{iz} = \cos z + i \sin z$

using contour:



or



Can similarly handle powers of trig functions $\cos^m x$, $\sin^m x$ by applying identities to reduce to (linear) combination of $\cos kx$, $\sin kx$, then do change of vars $x \to \frac{1}{k} x$.

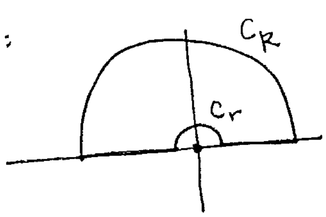
e.g. $\cos^2 x = \frac{1 + \cos 2x}{2}$.

Issue: All above assumes ~~that~~ poles of f do not lie on contour γ_R, or rectangle.

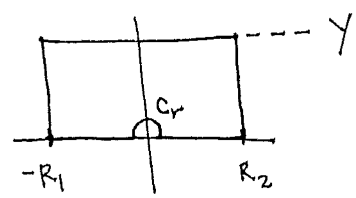
Example: $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}$. $\quad$ ← not defined at $x = 0$, but let $f(0) = 1 = \lim_{x \to 0} \frac{\sin x}{x}$

Again, no direct way ~~not easy~~ to estimate trig functions on circle. Better to consider $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iz}}{z}\, dz$

with contour:



or



Same estimates as before show C_R integral $\to 0$ as $R \to \infty$.

$$\lim_{r \to 0} \left[\int_{-\infty}^{-r} \frac{e^{ix}}{x}\, dx + \int_{r}^{\infty} \frac{e^{ix}}{x}\, dx \right] = \lim_{r \to 0} \int_{C_r} e^{iz}/z \, dz$$

since there are no residues inside contour γ_R, so by Cauchy's theorem $= 0$.

$\frac{e^{iz}-1}{z}$ has removable singularity at $z=0$ $\Rightarrow$ $\exists$ const. $M>0$ s.t.

$\left| \frac{e^{iz}-1}{z} \right| \leq M$ for $|z| \leq 1$

so $$\left| \int_{C_r} \frac{e^{iz}-1}{z}\, dz \right| \leq \pi \cdot r \cdot M \to 0$$

as $r \to 0$.

However $\int_{C_r} \frac{1}{z}\, dz = -\pi i$ (by direct computation) for all r

note: traversing in "opposite" direction

$$\Rightarrow \int \frac{e^{iz}}{z} = -\pi i.$$

Ahlfors (better): Use Laurent expansion.

$$\frac{e^{iz}}{z} = \frac{1}{z} + \underbrace{i + \cdots}_{\phi(z)}$$

analytic at origin.

still need to compute this directly.

Hence $$+\pi i = \underbrace{\int_{r}^{\infty} + \int_{-\infty}^{-r} \frac{e^{ix}}{x}\, dx}$$

$$e^{ix} = \cos x + i \sin x$$

Take imaginary parts of both sides. (or note $\frac{\cos x}{x}$ is odd)

Conclusion: $$\pi = \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$$

(note that limit we were computing was special since contour leaves vs evaluating at same points $r, -r$)

so we are really only concluding convergence in sense of Cauchy. However, this is ok since $\frac{\sin x}{x}$ even.)

Similar for $e^{iz} \cdot \frac{g(z)}{h(z)}$ with

$h(z)$ having real zeros.

$\deg h - \deg g \geq 1$.

Ex. 2 : $\displaystyle\int_0^\infty \frac{x^{-c}}{1+x}\,dx \qquad 0<c<1.$

$\dfrac{x^k}{x(1+x)} \quad 0<k<1$

Ahlfors notes

$x^k \cdot \dfrac{g(x)}{h(x)}$ must ~~have $h(x)$ with zero of order 2 at~~ $\deg h - \deg g \geq 2$ (at most) and g/h has simple pole at origin. for convergence.

As cx. function, z^{-c} not single valued. on $\mathbb{C}$

Define branch of z^{-c} by choosing branch of logarithm.

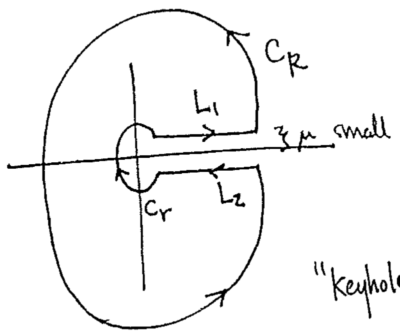
$$\log(re^{i\theta}) := \log r + i\theta \quad \text{on}$$

$$\Omega = \{z \mid z \neq 0 \text{ and } \arg(z) \in (0, 2\pi)\}$$

Then for $z \in \Omega$, set

$$z^{-c} := \exp(-c\log(z)).$$

γ_R :



"keyhole contour"

$$\int_{\gamma_R} f(z)\,dz = 2\pi i \cdot \operatorname{Res}_{z=-1} f$$

$$= 2\pi i \underbrace{(-1)^{-c}}_{\text{means}}$$

$\exp(-c \underbrace{\log(-1)}_{0+\pi i})$

$$= 2\pi i \exp(-c\pi i)$$

via Residue thm since $\gamma_R \sim 0$ in $\mathbb{C}$.

$$\int_{L_1} f(z) = \int_r^R \frac{(t+i\mu)^{-c}}{1+t+i\mu}\,dt$$

Want to show $\int_{L_1} \to \int$ we want

Define $g(t,\mu) : [r,R] \times [0, \pi/2]$

by $g(t,\mu) = \begin{cases} \left| \dfrac{(t+i\mu)^{-c}}{1+t+i\mu} - \dfrac{t^{-c}}{1+t} \right| & \mu > 0 \\ 0 & \mu = 0 \end{cases}$

g continuous, so uniformly continuous as defined on compact set.

Thus, given $\epsilon > 0$, $\exists\ \delta$ s.t. $|(t, \mu) - (t_0, \mu_0)| < \delta$

then $|g(t,\mu) - g(t_0,\mu_0)| < \epsilon/R$ Pick $\mu_0 = 0$, t_0 close to t (δ indep. of t_0, μ_0)

Gives $|g(t,\mu)| < \epsilon/R$

for $t \in [r, R]$

$\mu < \delta$

Hence
$$\int_r^R g(t,\mu)\, dt < \epsilon$$

for $\mu < \delta$. Since ϵ arb., $g(t,\mu)$ non-neg. then integral $= 0$.

$$\Rightarrow \int_r^R \frac{t^{-c}}{1+t}\, dt = \lim_{\mu \to 0^+} \int_{L_1} f(z)\, dz$$

Similarly,
$$\lim_{\mu \to 0^+} \int_{L_2} f(z)\, dz = -e^{-2\pi i c} \int_r^R \frac{t^{-c}}{1+t}\, dt$$

(use for example that $\log(\bar{z}) = \overline{\log(z)} + 2\pi i$)
with our definition of log.

<u>conclusion:</u>
$$\lim_{\mu \to 0^+} \left(\int_{L_1} + \int_{L_2} f\, dz \right) = \left(1 - e^{-2\pi i c}\right) \int_r^R \frac{x^{-c}}{1+x}\, dx$$

And by Residue thm, LHS $=$ $2\pi i \left(e^{-i\pi c}\right) - \left[\lim_{\mu \to 0^+} \int_{C_r} + \int_{C_R} f\, dz \right]$

Easy to see integrals over circles $\to 0$ as $r \to 0$, $R \to \infty$. E.g. for C_r:

$$\left| \int_{C_r} f \, dz \right| \le \int_{C_r} \frac{r^{-c}}{1-r} |dz| = \frac{r^{-c}}{1-r} \cdot 2\pi r \to 0$$

as $r \to \infty$

since $\deg_{\text{in } r}(\text{denom}) > \deg(\text{num in } r)$

Conclusion: (result of limits as $r \to 0$, $R \to \infty$)

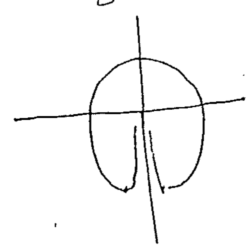
$$\int_0^\infty \frac{x^{-c}}{1+x} dx = \frac{2\pi i \, e^{-i\pi c}}{\underbrace{1 - e^{-2\pi i c}}} = \boxed{\frac{\pi}{\sin \pi c}}$$

$$\frac{2\pi i}{e^{+i\pi c} - e^{-\pi i c}}$$

Example 3: $\displaystyle\int_0^\infty \frac{\log x}{1+x^2} dx$

Avoid singularity with good choice of principal branch

Use branch of logarithm $\Omega = \{ z \in \mathbb{C} \mid z \neq 0. \ -\pi/2 < \arg z < 3\pi/2 \}$



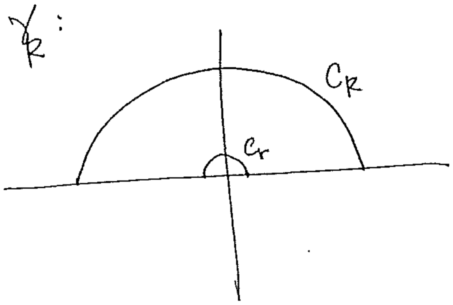
define $\log(z) := \log|z| + i\theta$

for $\theta \in (-\pi/2, 3\pi/2)$

so $\log(x)$ is as usual for $x > 0$ (real)

and $\log(x) = \log|x| + \pi i$ for $x < 0$.

γ_R:



$$\int_{\gamma_R} \frac{\log z}{1+z^2} dz = \int_r^R \frac{\log x}{1+x^2} + \int_{-R}^{-r} \left(\frac{\log|x| + \pi i}{1+x^2} \right) dx + \int_{C_R} f(z) \, dz + \int_{C_r} f(z) \, dz$$

By residue thm, since only pole in γ_R is at $+i$, with

residue $\dfrac{\log(i)}{2i} = \dfrac{\log|i|}{2i} + \dfrac{\pi i}{2(2i)} = \pi/4.$

so

$$\int_{\gamma_R} f(z)\,dz = \frac{\pi^2 i}{2}.$$

Moreover $\displaystyle\int_r^R \frac{\log x}{1+x^2}\,dx + \int_{-R}^{-r} \frac{\log x}{1+x^2}\,dx = 2\cdot\int_r^R \frac{\log x}{1+x^2}\,dx + \underbrace{\pi i\int_r^R \frac{dx}{1+x^2}}_{\to \pi/2 \text{ as } R\to\infty,\ r\to 0}$

Taking $r\to 0$, $R\to\infty$, not hard to show

C_R, C_r integrals $\to 0$.