Gamma function (king of all special functions)

working definition: functions that appear often.
just as useless as notion: "elementary function"

Better to try to characterize them.

Euler. want to ~~get~~ interpolate the discretely valued function $n!$ to a continuous real-valued or cx-valued function. Use power series or integral rep'n.

$$\int_0^\infty e^{-t}\, t^n\, dt = n! \quad \text{(integration by parts } n \text{ times)}$$

So use $\displaystyle\int_0^\infty e^{-t}\, t^s\, \frac{dt}{t}$ (why this funny normalization? discuss Haar measure for topological gps)

$$\Gamma(s) \quad \text{with} \quad \Gamma(n+1) = n!$$

Since, in general by parts,
$\Gamma(s+1) = s\cdot\Gamma(s)$

expression defines cx. analytic function for $Re(s) > 0$.

restriction comes from nbhd. of 0.
$|\text{integrand}| \sim t^{Re(s)-1}$
whose anti-deriv is $t^{Re(s)}$.

Play around with identities for $\Gamma(s)$.

$$\Gamma(s)\Gamma(1-s) = \int\int \cdots \quad (\text{makes sense for } Re(s) \in (0,1))$$

interchange order of integration. evaluate inner integral via change of vars

$$\ldots \quad \Gamma(s)\Gamma(1-s) = \int_0^\infty \frac{t^{1-s}}{1+t}\,\frac{dt}{t} = \int_0^\infty \frac{t^{-s}}{1+t}\, dt = \frac{\pi}{\sin \pi s}$$

$0 < Re(s) < 1$.

Our earlier result from contour integration/res. thm.

$$\Gamma(\tfrac12)^2 = \frac{\pi}{\sin\frac{\pi}{2}} \quad \text{i.e.} \quad \Gamma(\tfrac12) = \sqrt{\pi}.$$

Details of interchange of integration:

$$\Gamma(s)\Gamma(1-s) = \int_{\mathbb{R}^\times_{>0}} \int_{\mathbb{R}^\times_{>0}} u^s \, v^{1-s} \, e^{-u} \, e^{-v} \, \frac{dv}{v} \, \frac{du}{u}$$

$v \mapsto uv$

$$= \int_{\mathbb{R}^\times_{>0}} \int_{\mathbb{R}^\times_{>0}} u \cdot v^{1-s} \cdot e^{-u(1+v)} \, \frac{dv}{v} \, \frac{du}{u}$$

interchange integration

$u \mapsto u/_{1+v}$

$$= \int_{\mathbb{R}^\times_{>0}} \int_{\mathbb{R}^\times_{>0}} \frac{u}{1+v} \cdot v^{1-s} \, e^{-u} \, \frac{du}{u} \, \frac{dv}{v}$$

evaluate inner int : $\displaystyle\int_0^\infty e^{-u} \, du = 1$

$$= \int_{\mathbb{R}^\times_{>0}} \frac{v^{1-s}}{1+v} \, \frac{dv}{v}.$$

Where does Gamma appear?

- functions of exponential decay common in integral transforms / prob. distributions
  so can relate them to Gamma function by change of vars.

- when expressed as infinite product, has interesting set of poles.

- What is volume of n-sphere?

  $n=1: \quad 2r$

  $n=2: \quad \pi r^2$

  $n=3: \quad \frac{4}{3}\pi r^3.$

  n arbitrary:

  $$\pi^{n/2} \Big/ \Gamma(n/2+1) \cdot r^n$$

  ($1/2$-integers are related to $\sqrt{\pi}$,
  but other rationals not expected
  to be related to known transcendentals)

  $$n=3: \quad \frac{\pi^{3/2}\cdot r^3}{\Gamma(5/2)}$$

  $$\Gamma(5/2) = 3/2 \cdot \Gamma(3/2) = 3/2 \cdot 1/2 \cdot \Gamma(1/2) = 3/4 \cdot \sqrt{\pi}.$$

- "elliptic integrals" – integrals that arise in arc length of ellipse

  $$\int_0^1 \frac{dt}{\sqrt{1-t^3}} = \frac{\Gamma(1/3)^3}{\sqrt{3}\,\sqrt[3]{16}\,\pi}$$

  etc.

- theory of Riemann zeta function.