1'

$$\frac{\pi^2}{\sin^2 \pi z} = \pi^2 \csc^2 \pi z = \frac{d}{dz}\left(-\pi \cot \pi z\right)$$

Then applying Mittag-Leffler thm, to

$$\pi \cot \pi z = \sum_{n \in \mathbb{Z}} \Big[\underbrace{\frac{1}{z-n} - p_n(z)}_{} \Big] + g(z)$$

abs. conv. series, conv. unif. on compact sets

so can take derivatives termwise.

We draw the conclusion: $p_n(z)$, $g(z)$ constant.

so $p_n(z) = -\frac{1}{n}$, const. term in Taylor expansion of $\frac{1}{z-n}$ $(n \neq 0)$ at $z=0$

$p_0(z) = 0$.

Then can check directly that $\displaystyle\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \left(\frac{1}{z-n} + \frac{1}{n}\right) = \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{z}{n(z-n)}$

absolutely convergent by comparison to $\sum \frac{1}{n^2}$.

unif. on compact set.

$g(z) = 0$ because, grouping n and $-n$ terms together (which is permissible since series converges absolutely, so sum is not altered by regrouping)

then get

$$\underbrace{\pi \cot \pi z}_{\text{odd}} = \underbrace{\frac{1}{z}}_{\text{odd}} + \underbrace{\sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}}_{\text{odd}} + \underbrace{g(z)}_{\text{const.}} \Rightarrow g(z) = 0.$$

Example 2: $\frac{\pi}{\sin \pi z}$. Then singular parts are $\frac{(-1)^n}{(z-n)}$. Doesn't converge absolutely, but can prove

$$\frac{\pi}{\sin \pi z} = \lim_{m \to \infty} \sum_{-m}^{m} \frac{(-1)^n}{(z-n)}$$

via cotangent ident:

We have addressed poles of meromorphic functions. What about zeros?

Even for finitely many zeros, is there a canonical rep'n of all such functions?

If f is entire, never zero, then we can write

$$f(z) = e^{g(z)} \qquad g(z) : \text{entire.}$$

pf: $\frac{f'}{f}$ is analytic in whole plane, so is the derivative of entire function. call it $g(z)$.

$\Rightarrow$ $f(z)\, e^{-g(z)}$ has derivative identically 0

$\Rightarrow$ $f(z) = c \cdot e^{+g(z)}$ and can absorb constant c into $g(z)$. //

Then if $a_1, \dots, a_N$: zeros (repeated according to multiplicity) then we can write:

0 : zero of order m

$$f(z) = z^m e^{g(z)} \cdot \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right)$$

Naive guess: if $\{a_i\}$: possibly infinite collection of zeros of function, $a_i \neq 0 \ \forall i$

then write $$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)$$

What does it mean for infinite product to converge?

Ans: Just like for sums, consider partial products

$$\prod_{n=1}^{\infty} p_n := \lim_{n \to \infty} P_n \quad \text{where } P_n = p_1 \cdots p_n \, . \quad (\text{provided limit exists and } \neq 0)$$

(Note 0 bad because then if any $p_n = 0$, product converges independent of growth of factors)

Still too restrictive: Remove the (at most finitely many) factors $p_n = 0$ and ask that ~~[illegible]~~ sequence of partial prods formed from remaining p_n.

if partial products P_n converge, then $\lim_{n\to\infty} P_n / P_{n-1} = 1$

and since $P_n / P_{n-1} = p_n$, these $p_n \to 1$ as well.

Write products in form: $\prod_{n=1}^{\infty} (1 + b_n)$ with nec. condition for convergence $b_n \to 0$.

Key idea regarding convergence of infinite products: Take logarithm. Converts to infinite sum. Then assess convergence of sum.

$$\log\left(\prod_{n=1}^{\infty}(1+b_n)\right)$$

$$= \sum_{n=1}^{\infty} \log(1+b_n)$$ where we choose principal branch of logarithm here.

If $S_N := \sum_{n=1}^{N} \log(1+b_n)$ then $P_N = e^{S_N}$ and hence if

$S_N \to S$ then $P_N \to e^S$ $(\neq 0)$ (i.e. convergence of sum is sufficient for conv. of product)

In fact, convergence of sum of logs also necessary condition. (need to be careful about branch)

Suppose $P_N \to P \neq 0$. Then $\mathrm{Log}(P_N/P) \to 0$ as $N \to \infty$.

For any N, $\exists\, h_N$ s.t. $\mathrm{Log}(P_N/P) = S_N - \log P + h_N \cdot 2\pi i$ — sum won't converge to principal value. Just some value.

$\Rightarrow (h_{N+1} - h_N)\, 2\pi i = \mathrm{Log}(P_{N+1}/P) - \mathrm{Log}(P_N/P) - \mathrm{Log}(1 + a_N)$

Log: princ. branch with $\arg \in [-\pi, \pi]$

$\Rightarrow h_{N+1} = h_N$ for N suff. large, (call this integer h) since arg of rhs has $|\arg(1+a_N)| \leq \pi$ and $\arg(P_{N+1}/P) - \arg(P_N/P) \to 0$

Conclusion: $S_N \to \log P - h \cdot 2\pi i$

where h as in: $\log(P_N/P) = S_N - \log P + h\, 2\pi i$, $N \gg 0$.

To summarize :

$\prod_{n=1}^{\infty}(1+a_n)$ (where we assume $1+a_n \neq 0$) converges if and only if

$\sum_{n=1}^{\infty} \log(1+a_n)$ converges (where summands represent principle branch of log.)

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for absolute convergence, even simpler since $\sum_{n=1}^{\infty} |\log(1+a_n)|$ converges iff $\sum_{n=1}^{\infty} |a_n|$ converges

If (either of these) series converge absolutely, we say

$\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely.

(think Taylor expansions.

In particular $\lim_{z\to 0} \frac{\log(1+z)}{z} = 1$

i.e. since $|a_n| \to 0$ if (either) series converges, then the limit says for any $\epsilon > 0$ $\exists$ N s.t. $(1-\epsilon)|a_n| < |\log(1+a_n)| < (1+\epsilon)|a_n|$ for all $n \geq N$.)

which gives simultaneous convergence.

Similar equivalences are true for uniform convergence on compact sets, between products and corresponding sums

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Back to our original question :

How to make sense of :

$$f(z) = z^m e^{g(z)} \cdot \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right) \quad ?$$

$\{|a_n|\} \to \infty$

so $|z/a_n| \to 0$.

converges absolutely if and only if

$\sum_{n=1}^{\infty} |z/a_n|$ converges, i.e. if $\sum_{n=1}^{\infty} |1/a_n|$ converges ← otherwise need a correction...

(and then convergence is also uniform on compact sets: closed disks of radius R.)