Last week: meromorphic functions with prescribed zeros / poles (+ singular parts)

for poles: Mittag-Leffler Thm $\{a_n\}$ with $|a_n| \to \infty$ as $n \to \infty$

S_n: polys with no const. term

↖ then there exists f, merom, with poles $\{a_n\}$, singular parts S_n

Any such f is of form:

$$\sum_n \left(S_n\left(\frac{1}{z-a_n}\right) + p_n(z) \right) + g(z)$$

for some polynomials p_n, and g analytic (entire), and such that series conv. absolutely

Hidden: there aren't many good choices p_n, so that series converges absolutely.

Best choice most natural: p_n: a Taylor polynomial for $S_n\left(\frac{1}{z-a_n}\right)$ of sufficiently high degree

⇜ any other choice $p_n^*(z)$ differs by abs. conv. series which can be absorbed into the g.

Example: $\pi \cot \pi z = \sum_n \left(\frac{1}{z-n} + \frac{1}{n} \right)$

(p_n: const. terms of Taylor exp. of $1/z-n$

$g = 0$)

For zeros: Use products not sums.

Fix naive guess: $\{a_n\}$ with $|a_n| \to \infty$

Any such f with zeros at $\{a_n\}$ is of form: (f: analytic)

$$f(z) = e^{g(z)} z^m \prod_n \left(1 - \frac{z}{a_n}\right)$$

for some g ~~ent~~ analytic.

Just as with sums, correct terms in product so it absolutely converges.

To summarize :

$$\prod_{n=1}^{\infty} (1+a_n)$$ (where we assume $1+a_n \neq 0$) converges if and only if

$$\sum_{n=1}^{\infty} \log(1+a_n)$$ converges (where summands represent principle branch of log.)

For absolute convergence, even simpler since $\sum_{n=1}^{\infty} |\log(1+a_n)|$ converges iff $\sum_{n=1}^{\infty} |a_n|$ converges

(think Taylor expansions.

In particular $\lim_{z\to 0} \frac{\log(1+z)}{z} = 1$

i.e. since $|a_n| \to 0$ if (either) series converges, then the limit says

for any $\epsilon > 0$ $\exists$ N s.t. $(1-\epsilon)|a_n| < |\log(1+a_n)| < (1+\epsilon)|a_n|$ for all $n \geq N$.)

which gives simultaneous convergence.

If (either of these) series converge absolutely, we say

$$\prod_{n=1}^{\infty} (1+a_n)$$ converges absolutely.

Similar equivalences are true for uniform convergence on compact sets, between products and corresponding sums

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Back to our orginal question :

How to make sense of :

$$f(z) = z^m e^{g(z)} \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \quad ?$$

$\{|a_n|\} \to \infty$ so $|z/a_n| \to 0$.

converges absolutely if and only if

$$\sum_{n=1}^{\infty} |z/a_n|$$ converges, i.e. if $$\sum_{n=1}^{\infty} |1/a_n|$$ converges

otherwise need a correction...

(and then convergence is also uniform on compact sets.)

closed disks of radius R.

For poles, we had modified the singular part:

$$\sum_i P_i\left(\frac{1}{z-a_i}\right) - p_i(z) \quad \text{by} \quad p_i(z) : \text{Taylor polynomials for } P_i\left(\frac{1}{z-a_i}\right) \text{ at origin.}$$

Here, given $\{a_n\}$ with $|a_n| \to \infty$ as $n \to \infty$,

we ask for polynomials $p_n(z)$ such that

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{p_n(z)} \quad \text{converges to entire function.}$$

Key: by encoding in exponential, we ensure that no new zeros are introduced

(or equivalently, so that $\sum_n \left[\log\left(1-\frac{z}{a_n}\right) + p_n(z)\right]$ converges)

So take $p_n(z)$ to be the m_n-th Taylor poly of $\log\left(1-\frac{z}{a_n}\right)$ for $m_n \gg 0$.

Essentially a repeat of our earlier arg.

Thm (Weierstrass): $\exists$ entire function with arbitrarily prescribed zeros $\{a_n\}$ provided $|a_n| \to \infty$ as $n \to \infty$ (if sequence infinite)

Every such function is of form:

$$f(z) = z^m e^{g(z)} \cdot \prod_n \left(1 - \frac{z}{a_n}\right) e^{\overbrace{\frac{z}{a_n} + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}}^{-\, m_n\text{-th Taylor poly for } \log\left(1-\frac{z}{a_n}\right) \text{ at origin}}}$$

m_n: chosen so that prod. converges to entire function.

Corollary: If g is meromorphic on Ω, then

then $g = h_1/h_2$ $\quad h_1, h_2$: analytic on Ω

proof: take f as above with a_i : poles of g. Then $f \cdot g$ is entire. //

In fact

$$\log\left(1-\frac{z}{a_n}\right) + p_n(z) = -\frac{1}{m_n+1}\cdot\left(\frac{z}{|a_n|}\right)^{m_n+1} + \text{higher order terms}$$

Fix R, consider only $|a_n| > R$ when analyzing convergence for $|z| \le R$

Then $$|r_n(z)| \le \frac{1}{m_n+1}\left(\frac{R}{|a_n|}\right)^{m_n+1}\cdot\left(1-\frac{R}{|a_n|}\right)^{-1}$$

so suffices to show $\displaystyle\sum_{n=1}^{\infty}\frac{1}{m_n+1}\left(\frac{R}{|a_n|}\right)^{m_n+1}$ converges. (*)

m_n can be chosen so that:

(choose $m_n = n$, for example)

Obtain geometric series. Note degree of $p_n(z)$, which we've been calling m_n, may vary with n. But in practice, can often obtain required convergence by choosing $m_n = h$: fixed const. indep. of n. (just as in our examples with Mittag-Leffler thm: $\frac{\pi^2}{\sin^2 \pi z}$, $\pi \cot \pi z$
no correction deg. 0 only.)

If so, then $R^h/_{h+1}$ may be removed from (*) so we require $\sum 1/|a_n|^{h+1}$ to converge

Let h be smallest such integer ($h=\infty$ allowed)

Then we have a canonical expression for the product:

$$\prod_n \left(1-\frac{z}{a_n}\right) e^{\left[z/a_n + \cdots + \frac{1}{h}(z/a_n)^h\right]}$$

and for the function:

$$f(z) = z^m e^{g(z)}\cdot\prod_n\left(1-z/a_n\right) e^{[\text{—"—}]}$$

Define genus(f) $= \max(\deg(g), h)$ if g may be taken to be polynomial.

Example: $\sin \pi z$ is zero at integers

Want smallest h such that $\sum 1/|a_n|^{h+1}$ converges

i.e. $\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} 1/n^{h+1}$ converges. Then $h = 1$.

deg $h = 1$

So our canonical product takes form: $\sin \pi z = z e^{g(z)} \cdot \prod_{n \neq 0} (1 - z/n) e^{z/n}$

To determine genus, we need to know $\deg(g)$.

answer

Take logarithmic derivative on both sides (justified by uniform convergence of product on compact sets)

$$\pi \cot \pi z = \frac{1}{z} + g'(z) + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right)$$

But from our earlier Mittag-Leffler formula, we know $g'(z) = 0$

$\Rightarrow$ $g(z)$ constant.

Since $\lim_{z \to 0} \frac{\sin \pi z}{z} = \pi$, then $e^{g(0)} = \pi$

So $\sin \pi z = \pi z \cdot \prod_{n \neq 0} (1 - z/n) e^{z/n}$. (genus 1).

genus telling us about growth of function.

$$\text{order}(f) = \overline{\lim_{R \to \infty}} \frac{\log \log M(R)}{\log R}$$

$M(R)$: max of f on circle of radius R centered at origin.

then $h \le \text{order}(f) \le h+1$. h : genus