Exploring examples of differentiable functions:

Prove differentiability of nice classes of functions directly from the definition – e.g. polynomials, rational functions, power series.

For power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R

with $\frac{1}{R} = \limsup_n |a_n|^{1/n}$ s.t. for $|z| < R$, $f(z)$ converges absolutely, differentiable, with derivative given by term-by-term differentiation:

$$f'(z) = \sum_{n=1}^{\infty} n \cdot a_n z^{n-1}$$ (with same radius of conv. R)

Can REPEAT this process, proving power series are inf. diff. for $|z| < R$ with

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n z^{n-k}$$

$$= k!\, a_k + \cdots$$

so $f^{(k)}(0) = k!\, a_k \iff a_k = f^{(k)}(0)/k!$ and hence

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n .$$

CAUTION: Haven't proved anything about arbitrary analytic functions, only those initially defined as a power series.

Where do (important examples of analytic functions via) power series come from?

Answer: Solutions to linear ODEs. E.g. of the form

$$a_n(z) f^{(n)}(z) + \cdots + a_1(z) f^{(1)}(z) + a_0(z) f(z) = 0$$

(let's assume $a_n(z) \neq 0$ for simplicity) with or without initial conditions.

Substitute using $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and solve via recursive relations for a_n's.

Example $f(z) = f'(z)$ with initial condition $f(0) = 1$

then $a_0 + a_1 z + a_2 z^2 + \cdots = a_1 + 2a_2 z + 3a_3 z^2 + \cdots$

$$\Rightarrow \left.\begin{aligned} a_{n-1} &= n \cdot a_n \\ a_0 &= a_1 = 1 \quad \text{(from initial cond.)} \end{aligned}\right\} \quad a_n = \frac{1}{n!}$$

So solution: $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} =: e^z$ (owing to its agreement with e^x for x real.)

Questions: ① Where does it converge? What is R?

② What are its properties? Slightly tricky since defined as infinite series.

(not initially defining e as limit, or using inverse of logs, etc.)

For convergence, we compute R via

$$\limsup_{n\to\infty} \left|\frac{1}{n!}\right|^{1/n} = 0 \qquad \text{so} \quad R = \frac{1}{0} = \infty$$

(we are using root test to determine convergence. Can also use ratio test, proving that resulting R is same for root test since we already established all properties of power series from root test)

Prove limit by crude estimate for $n!$ e.g. show $n! > \left(\frac{n}{4}\right)^n$, for n suff. large.

In any case $R = \infty$, so e^z converges for all $z \in \mathbb{C}$.

Properties: (i) Additivity $e^a e^b = e^{a+b}$ $\forall\, a,b \in \mathbb{C}$

pf 1: Use thm. of Cauchy on multiplication of abs. convergent power series (Whittaker-Watson 2-53)

$$(1 + a + a^2/2 + \cdots)(1 + b + b^2/2 + \cdots) = 1 + (a+b) + \underbrace{\left(\frac{a^2}{2} + ab + \frac{b^2}{2}\right)}_{(a+b)^2/2} + \cdots$$

pf. 2: (prettier) Use differential equation + product rule. Given any $c \in \mathbb{C}$

$$D(e^z \cdot e^{c-z}) = e^z \cdot e^{c-z} + e^z \cdot (-e^{c-z}) = 0. \quad \forall\, z \in \mathbb{C}$$

claim: if $f'(z) = 0$ $\forall\, z \in \mathbb{C}$, then $f(z)$ constant.

pf of claim: $f = u + iv$ approach along real, imag paths.

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \equiv 0, \quad -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \equiv 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$$

all identically 0.

Using thm. from real-var. calculus, u, v constant on every horizontal, vertical line, hence constant on $\mathbb{C}$.

---

so $e^z \cdot e^{c-z} =$ constant. Setting $z = 0$, this is e^c.

Let $z = a$, $c = a+b$ so result follows.

Corollaries: (I) $e^z \cdot e^{-z} = 1 \Rightarrow e^z$ never 0.

(II) e^z has real coeffs in power series $\Rightarrow \overline{e^z} = e^{(\bar{z})}$

so $|e^z|^2 = e^z\,\overline{e^z} = e^{z+\bar{z}}$. If $z = iy$, then

$\Rightarrow |e^{iy}|^2 = 1$, that is, $|e^{iy}| = 1$ ← Try to understand e^{iy} more precisely.

---

e^z and trigonometric functions:

Define $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

Motivation: power series for sine, cosine match their real counterparts.

e.g. $\cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \cdots$

A little algebra from these definitions shows that :

1. $e^{iz} = \cos z + i \sin z$, $\forall\ z \in \mathbb{C}$

In particular: $e^{iy} = \cos y + i \sin y$ if $y \in \mathbb{R}$

(By uniqueness of power series rep'n , $\cos y$ & $\sin y$ are our familiar functions of real variable with geometric interpretation. Important that their definitions as cx. functions make no use of geometry.)

2. $\cos^2 z + \sin^2 z = 1$ $\forall\ z \in \mathbb{C}$

3. $D(\sin z) = \cos z$
$D(\cos z) = -\sin z$ (term-by-term. diff. of power series)

4. Other trig functions are thus rational functions in e^{iz}.

e.g. $\tan z = \dfrac{\cos z}{\sin z} = -i\left(\dfrac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right)$

5. Additivity of e^z gives addition formulas for sine/cosine

e.g. $\sin(a+b) = \cos a \sin b + \sin a \cos b$.

Final aside : Try other examples of linear ODEs :

Legendre Equation : $(1-z^2)\, f''(z) - 2z\, f'(z) + \alpha(\alpha+1) f(z) = 0$

α : parameter, $z \neq \pm 1$.

If α non-neg. integer, result is Legendre polynomials.

(ODE is natural because it arises from studying Laplace equation in spherical coordinates)

Any benefit to studying Legendre polynomials as functions of a cx. variable? As function of real-variable, interpretation as orthogonal polynomials.

Picking up on differential equations perspective, could also

define sin / cos as solutions to : $f''(z) + f(z) = 0$

(has a two dimensional space of solutions spanned by e^{iz}, e^{-iz}

which either follows from power series method of substitution or

just by noting e^z solves $f'(z) = f(z)$ plus chain rule)

To define two basis vectors for this space of solutions, might ask

for power series with real coefficients (or even/odd symmetry)

—

Similarly, solutions to $f''(z) = f(z)$ result in hyperbolic sine/cosine.

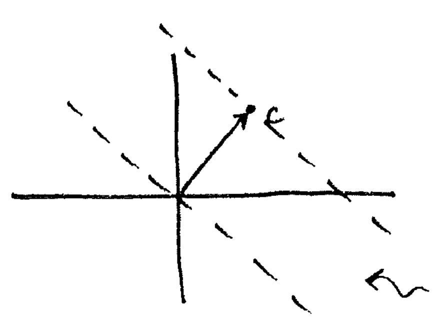
e.g. $\cosh z = \dfrac{e^z + e^{-z}}{2}$ (even sol'n).

—

Two important topics remaining on exponential function : ① periodicity ② inverse function

Definition : A function $f(z)$ is said to be periodic with period c if $f(z+c) = f(z) \quad \forall z \in \mathbb{C}$ $\quad (c \neq 0)$

In pictures,



just need to understand f on this strip (or any strip parallel to it of width $|c|$)

" fundamental domain " for $f(z)$

(Even more interesting : two periods c_1, c_2 linearly indep. over $\mathbb{R}$.) , Here fundamental domain is lattice in $\mathbb{R}^2 \leftrightarrow \mathbb{C}$

Notice that if $f(z+c) = f(z)$, then $f(z+2c) = f(z)$, etc
$\forall z \in \mathbb{C}$ so $f(z) = f(z+ck)$, $k \in \mathbb{Z}$.

Let's show e^z has a period. If c is a period for e^z, then by definition $e^{z+c} = e^z \quad \forall z \in \mathbb{C} \Rightarrow e^c = 1$.

We know that $|e^z| = e^x$ if $z = x+iy$, so c must be pure imaginary.

Hence we must find $y \in \mathbb{R}$ s.t. $e^{iy} = 1$.

(We know $e^{iy} = \cos y + i \sin y$, so want y such that $\cos y = 1$, $\sin y = 0$)

geometric intuition says $y = 2\pi k$, $k \in \mathbb{Z}$

Not enough for Ahlfors. What is π? Prove this analytically.

Step 1 : Show $\exists\ c$ such that $e^c = 1$.

(Use fact that $e^{iy} = \cos y + i \sin y$, then study them analytically. At the moment, we know $\cos z$, $\sin z$ are diff. functions with a few properties: derivatives, $\cos^2 z + \sin^2 z = 1$, addition laws)

First, we have basic estimate that

$\sin y < y$ if $y > 0$

(since $\sin y = y$ at $y = 0$ and $D(\sin y) = \cos y \leq 1$)

so can prove using integration.

Similarly $D(\cos y) = -\sin y \underset{\substack{\uparrow \\ \text{by estimate above}}}{>} -y$

and $\cos 0 = 1 \Rightarrow \cos y > 1 - \frac{y^2}{2}$ (via integration)

Using $\cos y > 1 - y^2/2$ $\forall y > 0$, then integrating $\int_0^y \cos t \, dt$, $\int_0^y (1 - \frac{t^2}{2}) dt$

We get $\sin y > y - y^3/6 \Rightarrow \cos y < 1 - y^2/2 + y^4/24$

Setting $y = \sqrt{3}$, we find $\cos \sqrt{3} < -1/8$, while $\cos 0 = 1$

$\Rightarrow$ $\exists\, y_0$ such that $\cos y_0 = 0$.
(in $(0, \sqrt{3})$)

Since $\cos^2 y_0 + \sin^2 y_0 = 1 \Rightarrow \sin y_0 = \pm 1 \Rightarrow e^{iy_0} = \pm i$

$\Rightarrow e^{4iy_0} = 1$. Conclusion: $(4y_0)i$ is a period!

Step 2: Show $4y_0 i$ is smallest period. (know pure imag. so we mean smaller than $4y_0$ as positive real.)

Suppose there were smaller, write it as $4y$ for some $y \in (0, y_0)$

Then $\sin y > 0$ (since $\sin y > y - y^3/6 = y \cdot (1 - y^2/6) > y/2 > 0$)
because $y < \sqrt{3}$.

$\Rightarrow \cos$ (strictly) decreasing on $(0, y_0)$

$\Rightarrow \sin$ (strictly) increasing on $(0, y_0)$ (since $\sin^2 z + \cos^2 z = 1$)

$\Rightarrow \sin y < \sin y_0 = 1$ so $0 < \sin y < 1$.

$\Rightarrow e^{iy} \neq \pm 1, \pm i \Rightarrow e^{i(4y)} \neq 1$ (contradiction)

Finally we arrive at the definition of π : $2\pi := 4y_0$.

(i.e. determined in terms of smallest period)

Along the way, we showed $e^{i\pi/2} = i$.

Step 3 : Show all periods are integer mults. of 2π.

If wi is another period, then we can find $k \in \mathbb{Z}$ s.t.

$2\pi(k) \leq w < 2\pi(k+1)$. If $w \neq 2\pi k$, then

$(w - 2\pi k)i$ is another positive period, contradicting the minimality of 2π.

Inverse functions : Try to define a function $z = \log w$

according to $w = e^z$.

Problems : ① $e^z \neq 0 \ \forall z \in \mathbb{C}$, so $\log(0)$ not defined.

② if $w \neq 0$, then $|w| = |e^z| = e^x$

so $e^{iy} = w/|w|$.

The equation $e^x = |w|$ has a unique solution $x = \log|w|$ (here : real logarithm)

But $e^{iy} = w/|w|$ on complex unit circle

has ∞-ly many solutions (differing by multiples of 2π)