Topology of Metric Spaces:

Let X be a metric space. S: subset of X. (viewed as metric space via same metric on X)

Definitions: Let $B(y,\delta)$ denote (open) ball of radius δ centered at y:

i.e. $B(y,\delta) := \{ x \in X \mid \underbrace{|x-y|}_{\text{i.e. } d(x,y)} < \delta \}$

Say N is a neighborhood of point $y \in X$ if $B(y,\delta) \subseteq N$ for some δ

A set Ω is open if it contains a neighborhood of each of its elements. (Trivial examples: ϕ, X: whole space)

A closed set is the complement of an open set.

We can define all of these terms relative to $S \subseteq X$ as well. Key change: Now $B(y,\delta)$ consists of points $x \in S$ (not all of X)

So set may be open in relative topology on S, but not open in X.

Example: S = unit circle $\subseteq \mathbb{C}$. Then S open in S, not in $\mathbb{C}$.

Final basic notion: accumulation point (or "limit point") of

$\overline{S}$: closure of S (w.r.t. X) $= \bigcap_{\substack{F \text{ closed in } X \\ S \subset F}} F$.

$x \in \overline{S}$ is accumulation point if every nbhd. of $x \in X$ contains ∞-ly many pts. of S.

Connectedness :

$S \subseteq X$ is connected if it cannot be expressed as

$S = A \cup B$, A, B disjoint, relatively open, (open in relative topology on S) non-empty sets.

Intuitively, "connected" means made of one piece.

Example : $S = \{ z \in \mathbb{C} \mid |z| \leq 1 \text{ or } |z-3| < 1 \}$: pair of disjoint circles in $\mathbb{C}$ of radius 1. (one w/ boundary one w/o.)

Is S disconnected ?

$A = \{ z \mid |z| \leq 1 \}$ $B = \{ z \mid |z-3| < 1 \}$ what if B defined by $|z-2| < 1$?

Key: A is open in relative topology.

(In fact if $S = A \cup B$ with A, B relatively open, then their complements B, A (respectively) are relatively closed. So equivalently S connected iff only subsets that are both relatively open, closed are ϕ, S.)

Thm : Connected subsets of the real line are intervals.

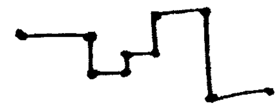
pf : See Ahlfors' Thm 1, p. 55.

Not so easy to characterize connected subsets of plane $\mathbb{R}^2 \simeq \mathbb{C}$ (as metric spaces)

But we'll finish with one very useful result about connected sets in plane:

Definition : A "rectilinear path" in plane consists of

$\sum_{i=1}^{k} [u_i, u_{i+1}]$ where $[u, v]$ denotes straight line path connecting $u, v \in \mathbb{C}$. We further require this line to ... parallel to an axis. I.e. vertical

Example in picture of rectilinear path: 

Thm: A non-empty open set Ω in plane is connected iff any two points can be joined by a rectilinear path which lies in Ω.

(Intuitively reasonable since open sets in plane have "thickness")

pf: ($\Rightarrow$) Assume Ω connected. Given $a \in \Omega$, we partition the set into $\Omega_a^+ \cup \Omega_a^-$

$$\Omega_a^+ := \{ z \in \Omega \mid z \text{ has rect. path to } a \text{ in } \Omega \}$$

$$\Omega_a^- := \Omega - \Omega_a^+ \quad \text{(i.e. pts. with no such path)}$$

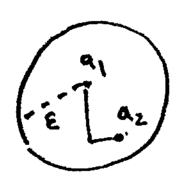
show that both sets open $\Rightarrow$ (since Ω assumed connected, $a \in \Omega_a^+ \neq \phi$) $\Omega_a^- = \phi$.

Ω_a^+ is open, since if $a_1 \in \Omega_a^+$, then $\exists$ open ball $B(a_1, \varepsilon) \subseteq \Omega$ (since Ω open)

If $a_2 \in B(a_1, \varepsilon)$, then $a_2 \in \Omega_a^+$. Indeed

a is connected to a_1 via rect. path., a_1 connected to a_2 via rect. path:

so a has rect. path to a_2 in Ω.



(contained in $B(a_1, \varepsilon) \subseteq \Omega$ by triangle ineq.)

By similar logic, Ω_a^- open since, if

$a_3 \in \Omega_a^-$, and $a_4 \in B(a_3, \varepsilon) \subseteq \Omega$ for some ε

then $a_4 \in \Omega_a^-$ (since a_3 and a_4 are joined by rect. path, so if a_4 had rect path to a, so would a_3. $\lightning$.)

pf of thm. (cont.) : ($\Leftarrow$) Proof by contrapositive.

Assume Ω not connected $\Rightarrow$ $\Omega = \Omega_1 \cup \Omega_2$ disjoint, non-empty open sets.

Show $\exists$ two points not connected by rect. path.

Pick $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$. If joined by rect. path, then $\exists$ straight line segment connecting a point a_1 in Ω_1 to a point a_2 in Ω_2.

So suffices to show this segment can't exist.

Let this segment be described by its parametric equation

$$L : \quad z = a_1 + t(a_2 - a_1) \qquad t \in [0,1].$$

$\Omega_1 \cap (L \text{ with } t \in (0,1))$ open*, non-empty
$\Omega_2 \cap (L \text{ with } t \in (0,1))$ — " —
} union is L with $t \in (0,1)$

this contradicts connectedness of $(0,1)$ in previous thm.

*: takes short argument to show open:

* claim: $\{ t \in (0,1) \mid L(t) = a_1 + t(a_2 - a_1) \in \Omega_1 \}$ is open.

Call this L_1. If $t_0 \in L_1(t)$, need to show $\exists$ ε-nbhd of t_0, i.e. $(t_0 - \varepsilon, t_0 + \varepsilon) \in L_1$

But Ω is open so $\exists$ open ball $B(a_1 + t_0(a_2 - a_1), \delta) \subseteq \Omega_1$ for some δ

Pick $\varepsilon < \dfrac{\delta}{|a_2 - a_1|}$.

Thus we have, sketched a proof that:

Ω open, connected in plane $\iff$ Any pair of points connected by rectilinear path

Corollary: If $f' \equiv 0$ for all points in open, connected set, then f is constant

pf: We proved this for $f' \equiv 0$ $\forall z \in \mathbb{C}$ and the same proof works:

choosing horiz./vert. paths $\Rightarrow$ $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \equiv 0$ on Ω

$\Rightarrow$ f const. on all horiz, vertical lines. $\Rightarrow$ f const. on Ω, since any two points are connected by rect. path.

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Compactness

Definition: $S \subseteq X$ compact iff every open cover of S admits a finite subcover. "Heine-Borel property"

(Intuition: "compact" means closed, bounded. Bounded: max distance between any two points bounded by fixed constant.)

we'll see momentarily that the intuitive def. is equivalent to the formal definition if $X = \mathbb{R}$ or $\mathbb{C}$.

However, the formal definition is very useful for proofs.

Example: S compact $\Rightarrow$ S complete (i.e. every Cauchy sequence converges)

Pf: If y is not limit of $\{x_n\}$, there exists $\epsilon > 0$ s.t.

$d(x_n, y) > 2\varepsilon$ for inf. many n. (Logical equiv. to definition of limit) pf cont. : So if $\{x_n\}$ Cauchy, then choose N s.t.

$d(x_m, x_n) < \varepsilon$ if $m, n \geq N$. $\Rightarrow$ $d(x_m, y) > \epsilon$ if $m \geq N$

conclusion : $B(y, \varepsilon)$ contains only finitely many of $\{x_n\}$ if $\{x_n\}$ Cauchy.

(since x_m, x_n close for $\forall\, m, n \geq N$ and $\exists\, n \geq N$ with x_n, y apart $\Rightarrow$ x_m, y apart $\forall\, m \geq N$.

Suppose S compact. $\{x_n\}$ Cauchy but doesn't converge. Then each $y \in S$ has $B(y, \varepsilon)$ with fin. many x_n, but $\{B(y, \varepsilon)\}_{y \in S}$ cover S.

So there is a finite subcover. ↯ since finite subcover contains only finitely many x_n. $\Rightarrow$ sequence is finite!

Example 2 : S compact $\Rightarrow$ S totally bounded

(Here totally bounded means, for every $\epsilon > 0$, S is covered by finitely many balls of radius ε.)

pf : clear. Just take all balls of radius ε : $\bigcup_{y \in S} B(y, \varepsilon) \supseteq S$, which admits finite subcover.

Note : Totally bounded $\Rightarrow$ bounded (Fix any ε. Let $x_1, \ldots, x_k$ be centers of balls in cover.

Then $d(x, y) < 2\varepsilon + \max_{i \in [1, k],\, j \in [1, k]} (d(x_i, x_j))$

Thm : S compact $\Leftrightarrow$ S complete, totally bounded.

(we have shown $\Rightarrow$. See p. 61 of Ahlfors for $\Leftarrow$)

So just have to show ($\Leftarrow$) ✓

Corollary : Subset of $\mathbb{R}$ or $\mathbb{C}$ is compact $\Leftrightarrow$ closed, bounded.

pf : compact $\Rightarrow$ complete $\Rightarrow$ closed. , compact $\Rightarrow$ tot. bounded $\Rightarrow$ bounded.

Cor. continued: For $(\Leftarrow)$, $\mathbb{R}, \mathbb{C}$ complete, so closed subsets are complete.

So just have to show totally bounded $\Leftrightarrow$ bounded (easy exercise)
$(\Rightarrow)$ already done
so just need $(\Leftarrow)$ //

Final equivalent notion of compactness:

Theorem (Bolzano-Weierstrass) S compact $\Leftrightarrow$ every infinite sequence contains limit point
(i.e. convergent subsequence)

Intuition over $\mathbb{R}$ or $\mathbb{C}$: closed, bounded subset

then infinite sequence must have points clustered together.

Pf: $(\Rightarrow)$ Same as showing compact $\Rightarrow$ complete.

if y not a limit pt. of $\{x_n\}$, then $\exists$ $B(y,\varepsilon)$ with finitely many x_n's.

No limit points $\Rightarrow$ $\bigcup_{y \in S} B(y,\varepsilon)$ is open cover $\underset{\text{(by comp.)}}{\Rightarrow}$ has finite subcover
$\Rightarrow$ sequence finite ↯.

$(\Leftarrow)$ Show S is complete, totally bounded.

Complete: every Cauchy sequence has limit pt., hence converges.

totally bounded: If not, $\exists$ $\epsilon > 0$ s.t. no cover by finitely many balls of radius ε.

Form sequence w/o limit point: pick x_1 randomly.
Pick x_n s.t. $x_n \notin B(x_1,\varepsilon) \cup \cdots \cup B(x_{n-1},\varepsilon)$
(can do this since $\bigcup_{i=1}^{n-1} B(x_i,\varepsilon)$ not an open cover). ↯.

Connectedness / Compactness are preserved by continuous functions:

Thm: If f continuous $X \to X'$ (metric spaces), S compact / connected (respectively)

then $f(S)$ compact (resp. connected)

pf: Recall that f is continuous iff inverse image of every open set is open (iff inverse image of every closed set is closed)

(just the ε-δ definition expressed in language of open sets)

① If f is continuous, S compact. $\bigcup_i \Omega_i$: open cover of $f(S)$

$f^{-1}(\Omega_i)$ are open cover of S, so have finite subcover, apply f.

② S connected. Suppose $S' = f(S) = A \cup B$ open, disjoint

Then $S = f^{-1}(A) \cup f^{-1}(B) \Rightarrow f^{-1}(A)$ or $f^{-1}(B)$ empty

$\Rightarrow A$ or B empty.

Nice application: $f : S \to \mathbb{R}$ continuous has $f(x)$ = interval if S connected

"intermediate value thm"

Final topic: uniform continuity. f is uniformly continuous on S if (say $f: S \to S'$)

for every $\epsilon > 0$, $\exists\ \delta > 0$ s.t. $d'(f(x_1), f(x_2)) < \epsilon$

for all pairs x_1, x_2 with $d(x_1, x_2) < \delta$. (i.e. δ independent of $x \in S$)

Thm: S compact. $f: S \to S'$ continuous, then f uniformly continuous.