Given function $\phi : \mathbb{R} \to \mathbb{C}$ or from $[a,b] \in \mathbb{R} \to \mathbb{C}$,

write $\phi(t) = u(t) + i\,v(t)$, $u, v : \mathbb{R} \to \mathbb{R}$, and

so define
$$\int_a^b \phi(t)\,dt \overset{\text{def}}{:=} \int_a^b u(t)\,dt + i \int_a^b v(t)\,dt \qquad (a, b \in \mathbb{R})$$

(provided both integrals on the right exist)

Can use this to define complex line integral.

Note: Basic properties of usual real integral hold for our version defined above:

(1) $c \in \mathbb{C}$, then $\displaystyle\int_a^b c \cdot \phi(t)\,dt = c \cdot \int_a^b \phi(t)\,dt$ (Easy check. using def'n)

(2) $$\left| \int_a^b \phi(t)\,dt \right| \le \int_a^b |\phi(t)|\,dt \quad \text{if } a \le b.$$

pf of (2) If $\displaystyle\int_a^b \phi(t)\,dt = 0$, then done. Else let

$\theta := \operatorname{Arg}\left(\int_a^b \phi(t)\,dt \right)$. Using property (1),

$$\underbrace{\operatorname{Re}\left[e^{-i\theta} \int_a^b \phi(t)\,dt \right]}_{= \left| \int_a^b \phi(t)\,dt \right|} = \int_a^b \operatorname{Re}\left[e^{-i\theta} \phi(t) \right] dt$$

$$\le \int_a^b |\phi(t)|\,dt$$

since $\operatorname{Re}\left(e^{-i\theta} \phi(t) \right) \le \left| e^{-i\theta} \phi(t) \right| = |\phi(t)|$

The data associated to complex line integral are:

1. connected, open set $\Omega \subseteq \mathbb{C}$.
2. continuous function $f : \Omega \to \mathbb{C}$
3. a smooth (i.e. continuously differentiable) path $\gamma : [a,b] \to \Omega$.

WARNING: Ahlfors calls this a "differentiable arc" even though he means "continuously differentiable". (cf. p. 68 for discussion)

Then define two types of line integrals:

$$\int_\gamma f(z)\, dz \overset{\text{def}}{=} \int_a^b \underbrace{f(\gamma(t))\, \gamma'(t)}_{\substack{\text{think of this as} \\ \phi(t) \text{ in earlier} \\ \text{discussion}}}\, dt$$

$$\int_\gamma f(z) \underbrace{|dz|}_{\text{sometimes "}ds\text{"}} \overset{\text{def}}{=} \int_a^b f(\gamma(t)) |\gamma'(t)|\, dt$$

Notation on left reflects desire that these integrals depend on path γ but not manner in which it is traversed (i.e. way of parametrizing γ). So is the RHS indep. of parametrization?

Suppose $t := t(\tau)$ is an increasing function mapping $\tau \in [\alpha, \beta]$ to $t \in [a,b]$, and t smooth. From theory of Riemann integral:

$$\int_a^b f(z(t))\, z'(t)\, dt = \int_\alpha^\beta f(z(t(\tau)))\, \underbrace{z'(t(\tau))\, t'(\tau)}_{\frac{d}{d\tau}(z(t(\tau)))}\, d\tau$$

(usual formula for change of vars.)

Note that we required $t := t(\tau)$ to be increasing. Indeed, line integrals depend on orientation of path. In particular, if $-\gamma$ is the same path but traversed in opposite direction (explicitly if γ given by $z = z(t)$, $t \in [a,b]$, then $-\gamma$ given by $z(-t)$ with $t \in [-b,-a]$)

then

$$\int_{-\gamma} f(z)\,dz = \int_{-b}^{-a} f(z(-t))\,(-z'(-t))\,dt$$

$$\underset{t \mapsto -t}{=} \int_{b}^{a} f(z(t))\,z'(t)\,dt = -\int_{a}^{b} f(z(t))\,z'(t)\,dt$$

$$= -\int_{\gamma} f(z)\,dz.$$

However, one sees that for the line integral w.r.t. arc length

$$\int_{\gamma} f(z)\,|dz| = \int_{-\gamma} f(z)\,|dz|.$$

Furthermore, clear that basic ineq.

$$\left| \int_{\gamma} f\,dz \right| \leq \int_{\gamma} |f|\,|dz|$$

follows from applying Property ②:

Remarks:

① All of this can be extended to piecewise smooth curves γ. These are composed of fin. many smooth curves and we just make same definition on each smooth piece.

② $$\text{length}(\gamma) \overset{\text{def}}{=} \int_{\gamma} |dz| \overset{\text{def}}{=} \int_{a}^{b} |\gamma'(t)|\,dt,$$ just as in case of real Riemann integral.

A more general (possibly better) definition of complex line integral:

Ω, f as before, but now $\gamma : [a,b] \to \Omega$ only assumed <u>continuous</u>

Define: $$\int_\gamma f(z)\,dz = \lim_{\text{mesh}(P)\to 0} \sum_{j=1}^{m} f(\gamma(t_j^*))\,(\gamma(t_j)-\gamma(t_{j-1}))$$

where limit is over partitions P of interval $[a,b]$ with $P = \{t_0 = a < t_1 < \dots < t_m = 1$

and sample point $t_j^* \in [t_{j-1}, t_j]$ with $\text{mesh}(P) := \max\{t_j - t_{j-1}\}$

"Riemann-Stieltjes integral". (Similar definition for $\int_\gamma f(z)\,|dz|$ with $\gamma(t_j)-\gamma(t_{j-1})$ replaced by $|\gamma(t_j)-\gamma(t_j$

<u>Existence Thm</u>: If $\gamma : [a,b] \to \Omega$ is of bounded variation,

then $\int_\gamma f(z)\,dz$ exists. (Bounded variation means $\exists$ const. M s.t., for any partition $P = \{t_j\}$, of $[a,b]$

$$V_P(\gamma) := \sum_{i=1}^{m} |\gamma(t_j)-\gamma(t_{j-1})| \le M\ )$$

pf of existence is somewhat painful. See p. 60-61 of Conway.*

In fact $\sup_P V_P(\gamma)$ can be shown to equal $\int_a^b |\gamma'(t)|\,dt$ $\overset{!!}{}$ length(γ)

pf of compatibility with earlier definition (in case γ smooth) not so bad. At least, fairly straightforward for $\int_\gamma f(z)\,dz$

Less so for $\int_\gamma f(z)|dz|$.

So equivalently, we require γ to have finite length ("rectifiable")

... existence of Riemann integral as well.

Example : $\gamma : [0, 2\pi] \longrightarrow C_r = \{ z \mid |z| = r \}$

$t \longmapsto r \cdot e^{it}$

$f_n(z) = z^n \quad (n \in \mathbb{Z})$, Ω : open, connected set containing C_r

(omit $z = 0$ if $n < 0$)

e.g. open annulus.

Then $$\int_\gamma f_n(z)\,dz = \int_0^{2\pi} (r \cdot e^{it})^n \frac{d}{dt}(re^{it})\,dt$$

$$= \int_0^{2\pi} r^n \cdot e^{itn} \cdot ir\,(e^{it})\,dt$$

$$= i\,r^{n+1} \cdot \int_0^{2\pi} \underbrace{e^{i(n+1)t}}_{=0 \text{ unless } n=-1}\,dt$$

express as $\cos(n+1)t + i\sin(n+1)t$ do integrals separately as real Riemannian integrals

$$= \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{otherwise.} \end{cases}$$

Next time we'll see how this example reflects more general result:

If f has an antiderivative F on Ω s.t. $F' = f$, then

$\int_\gamma f\,dz$ depends only on endpoints of γ.

A further preview: Suppose $f(z) = \sum_{-\infty}^{\infty} a_n z^n$. Then, provided we can pass integration through summation, we'd have $\frac{1}{2\pi i} \int_\gamma f(z)\,dz = a_{-1}$.

if γ : circle of radius r.

Compatibility of definitions of complex line integral:

Ω : open, connected $\subseteq \mathbb{C}$, $f : \Omega \to \mathbb{C}$ continuous. $\gamma : [a,b] \to \Omega$
smooth path

Definition 1 (Riemann int.)

$$\int_\gamma f\, dz = \int_a^b f(\gamma(t))\,\gamma'(t)\, dt$$

$$= \lim_{\text{mesh}(P)\to 0} \sum_j f(\gamma(t_j^*))\,\gamma'(t_j^*)\,(t_j - t_{j-1})$$

efinition 2 (Riemann-Stieltjes int.)

t_j^* : sample pts in $[t_{j-1}, t_j]$

$$\int_\gamma f\, dz = \lim_{\text{mesh}(P)\to 0} \sum_j f(\gamma(t_j^*))\,(\gamma(t_j) - \gamma(t_{j-1}))$$

Comparing summands, in definition 1, we have

$$\gamma'(t_j^*)\,(t_j - t_{j-1}) = (u'(t_j^*) + i\, v'(t_j^*))\,(t_j - t_{j-1}) \qquad (1)$$

for $\gamma = u + iv$,

via Mean Value thm (for real-valued functions)

$$\gamma(t_j) - \gamma(t_{j-1}) = (u'(s_j^*) + i\, v'(w_j^*)) \quad \text{some } s_j^*, w_j^* \text{ in } [t_{j-1}, t_j] \qquad (2)$$
$$\times (t_j - t_{j-1})$$

Since u', v' continuous on compact set (namely $[a,b]$), they are uniformly continuous.

so ~~[illegible]~~ can make difference between (1) and (2) as small as we like ($< \varepsilon'(t_j - t_{j-1})$) for partitions P with fine enough mesh.

Moreover $f(\gamma(t))$ is bounded since f continuous and $\text{Im}(\gamma)$ on $[a,b]$

Thus choose ε' so that the summand

$$\left| f(\gamma(t_j^*)) \; (u'(t_j^*) - u'(s_j^*) + i(v'(t_j^*) - v'(w_j^*)) \cdot (t_j - t_{j-1}) \right| < \frac{\varepsilon \, (t_j - t_{j-1})}{b-a}$$

This is enough because, summing over all summands, we get that the difference between the two definitions is $< \varepsilon$. //

$\Rightarrow$ integrals are equal.