8701 – Take Home Final – DUE: Thursday, Dec. 19, 5 PM

Instructions: Complete all of the following problems. You are free to use any notes or books, and expository resources from the web, but not allowed to collaborate with other students in the class or seek out specific solutions on the web. In solving the problems, feel free to quote any results in Ahlfors, but be sure to do so precisely. Each question is worth 10 points. Your solutions may be turned in to either my department mailbox, or emailed to me.

1. Prove that the function $f(z)$ defined by

$$f(z) = \begin{cases} e^{-1/z^4} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

satisfies the Cauchy-Riemann equations at all points in the plane, but nevertheless fails to be entire.

2. Compute

$$\int_\gamma \frac{dz}{z^2 - 1}$$

where γ is the semicircular boundary of $|z| = 2$ with $\mathrm{Re}(z) \geq 0$.

3.

a) Compute

$$\int_{|z|=1} \overline{z}\, dz$$

b) Let f be analytic on an open, connected set Ω with continuous derivative $f'(z)$ and let γ be a simple closed curve in Ω. Find

$$\mathrm{Re}\left(\int_\gamma \overline{f(z)} f'(z)\, dz \right).$$

4. Show that there is no function f, analytic in a neighborhood of a point z, such that

$$|f^{(n)}(z)| > n^n\, n! \qquad \text{for all positive integers } n$$

5. Compute the improper integral, for any real $a > 0$,

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)^2}\, dx$$

6. Show that the polynomial $f(z) = z^5 + 3z^3 + 7$ has all its zeros inside the open ball $|z| < 2$.

7.

a) Let f be analytic on the punctured unit disk $D\{0,1\} \setminus \{0\}$. Prove that if $\mathrm{Re}(f(z)) > 0$ on this region, then f has (at worst) a removable singularity at the origin.

b) Prove that there is no function $f(z)$ which is both analytic on the punctured unit disk $D\{0,1\} \setminus \{0\}$ and such that $f'(z)$ has a simple pole at the origin.

8. Exhibit a function that has poles at each positive integer n, each with order n, and is analytic and non-zero at every other complex number.

9. In one proof of the prime number theorem, the logarithmic derivative is inserted into an integral transform, whose evaluation is known as the Perron formula. In this problem, you are asked prove this formula, valid for any real $c > 0$:

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s} ds = \begin{cases} 0 & 0 < x < 1 \\ 1/2 & x = 1 \\ 1 & x > 1. \end{cases}$$

(The convergence in the case of $x = 1$ is only in the Cauchy sense.)