Math 8702 – Spring 2014 – Take Home Final Exam
DUE: Saturday, 5 pm, May 17

You have two choices for the final exam. You may either do a project on the uniformization theorem, described below, OR answer the following 5 questions. (Don't do both.)

Option 1: Traditional Take-Home Exam

1. Suppose that $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are non-constant holomorphic maps of Riemann surfaces. Show that the degree of $g \circ f$ is the product of the degrees of f and g.

2. Let X be the *compact* Riemann surface associated to the affine equation

$$z^{2a} - 2w^b z^a + 1 = 0, \quad a, b \text{ fixed positive integers.}$$

Determine the genus of X as a function of a and b.

3. Let $f(z) = 4z^2(z-1)^2/(2z-1)^2$. Then f defines a holomorphic map of degree 4 from the Riemann sphere to itself. Show there are three branch points and find the resulting monodromy representation.

4. Recall that Cauchy's theorem for Riemann surfaces states that if $S \subset X$ is a compact Riemann surface with boundary, and α is a holomorphic 1-form on a neighborhood of S, then

$$\int_{\partial S} \alpha = 0.$$

i Prove Cauchy's theorem.

ii Recall further that a meromorphic 1-form α on X is locally expressible as $f(z)dz$ where f is a meromorphic function. Use Cauchy's theorem to prove that the sum of the residues of a meromorphic 1-form on a compact Riemann surface X is zero.

5. Let $\{p_1, \ldots, p_n\}$ be distinct points on a compact Riemann surface X. Show that there exists a meromorphic function on X mapping $p_i = w_i$ for any choice of n distinct points $w_1, \ldots, w_n$ in $\mathbb{C}$.

Option 2: Project on Uniformization Theorem

The uniformization theorem describes the set of all simply-connected Riemann surfaces up to isomorphism. The write-up should include a statement of the theorem, a sketch of the proof, and a discussion of its implications for the classification of Riemann surfaces up to isomorphism.

That is, explain how Riemann surfaces arise from this classification together with a description of the holomorphic maps from a simply connected Riemann surface to itself. (What conditions must be placed on these maps to ensure the quotient is a Riemann surface?) Give some examples of Riemann surfaces that arise by such constructions.

I expect that a thorough write-up should take 5–7 typed pages, but could certainly be longer if the student really got engaged and wanted to do more. The uniformization theorem is in many books on Riemann surfaces, but not in Miranda. Donaldson's book (available on the web) discusses the result in Chapter 10 and it nicely follows the topics we covered in the last week of class.