So far

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{w \in M \\ w \neq 0}} \frac{1}{(z-w)^2} + \frac{1}{w^2}$$

(maybe call it $\wp_M$ to emphasize dependence on M.)

defines merom. function with double poles in M.

Now show $\wp(z)$ is elliptic.

pf: p. 176, thm 1 of Ahlfors is Weierstrass' result that uniform conv. on compacta $\Rightarrow$ series may be differentiated term by term.

So $$\wp'(z) = -2 \sum_{w \in M} \frac{1}{(z-w)^3}$$

Now clearly $z \mapsto z + w'$ for $w' \in M$ just permutes terms in sum over M.

$\Rightarrow \quad \wp'(z + w_i) = \wp'(z) \qquad w_i$ = one of generators of M

$\Rightarrow \quad \wp(z + w_i) = \wp(z) + c$ for some constant $c \in \mathbb{C}$

Plugging in $-w_i/2$, then we see $c = 0$.

Proposition: The elliptic functions w.r.t. M (always assuming meromorphic) form a field. It is generated by $\wp, \wp'$ — i.e. every such f is rational expression in $\wp, \wp'$ with coeffs in $\mathbb{C}$.

pf: Given any such f, write

$$f(z) = \underbrace{\frac{f(z) + f(-z)}{2}}_{\text{even}} + \underbrace{\frac{f(z) - f(-z)}{2}}_{\text{odd}}.$$

$\wp'(z)$ odd

$\wp(z)$ even.

If f odd, then $\wp' f$ is even, so ~~we may reduce to question of~~ it suffices to show that ~~showing~~ all <u>even</u>, elliptic f are expressible as rational functions in $\wp$. This is one of hw questions for next week (#1, p. 274)

Just need some rational expression in $\wp$ with same zeros/poles. Then $R(\wp)$

$f/R(\wp)$ is elliptic function w/o zeros or poles, hence constant.

Is there an algebraic relation between $\wp, \wp'$? Yes. to find it, study power series expansions at origin.

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in M \setminus \{0\}} \underbrace{\frac{1}{(z-\omega)^2}}_{\text{expand as geom. series}} - \frac{1}{\omega^2}$$

$$\left(\frac{1}{\omega^2} \left[\frac{1}{1-\frac{z}{\omega}} \right]^2 \right) = \frac{1}{\omega^2} \left[1 + \frac{z}{\omega} + \frac{z^2}{\omega^2} + \cdots \right]^2$$

so
$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in M \setminus \{0\}} \frac{1}{\omega^2} \sum_{n=1}^{\infty} (n+1) \left(\frac{z}{\omega}\right)^n$$

$$= \frac{1}{z^2} + \sum_{n=1}^{\infty} (n+1)\, z^n \cdot \underbrace{\sum_{\omega \in M \setminus \{0\}} \frac{1}{\omega^{n+2}}}_{G_{n+2}}$$

reverse order of summation

this is our Taylor expansion at $z=0$.

if n odd, then G_n is 0 since $\omega, -\omega$ give opposing contributions.

G_n is absolutely conv. if $n \geq 3$ from Wednesday's computation.

To find power series for $\wp'(z)$, differentiate term by term:

$$\wp'(z) = \frac{-2}{z^3} + \sum_{n=1}^{\infty} n\cdot(n+1)\, G_{n+2}\, z^{n-1}$$

$$= \frac{-2}{z^3} + 6G_4 z + 20 G_6 z^3 + \cdots$$ and we had:

$$\wp(z) = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + \cdots$$

So to find alg. relation, try to remove lowest terms: $\wp'(z)^2 - 4\wp(z)^3$

Keep going... Find $\wp'(z)^2 - 4\wp(z)^3 + 60G_4\,\wp(z) + 140\,G_6$ (*)

is elliptic function with no neg. powers in Laurent exp. at 0, hence no poles at 0

But already knew $\wp, \wp'$ were analytic away from M.

Thus this expression must be constant. Find constant by calculating constant term of power series for (*). It is 0.

$\Rightarrow$ one relation $\wp'(z)^2 = 4\wp(z)^3 - 60G_4\,\wp(z) - 140\,G_6$.

We can even factor right-hand side by considering 0's of $\wp'(z)$:

$\wp'$ is odd + elliptic, so will have zeros whenever $\wp'(a)$ defined and $a \equiv -a \mod M$.

$a \equiv -a \mod M$ at $a = \underset{\text{pole}}{0}, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ $\Rightarrow$ $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ zeros of $\wp'$

$\Rightarrow$ ——"—— doubled zeros of $\wp$

$\wp(\frac{\omega_1}{2}), \wp(\frac{\omega_2}{2}), \wp(\frac{\omega_1+\omega_2}{2})$ distinct cx. #'s, else one of $\wp(z) - \wp(\frac{\omega_i}{2})$ wouldn't have # zeros − # poles = 0 in fund. //-gram

This differential equation can be solved in the form of an identity:

$$\int_{\wp(z_0)}^{\wp(z)} \frac{dw}{\sqrt{\underbrace{4w^3 - 60G_4 w - 140G_6}_{4\left(w - \wp\left(\frac{\omega_1}{2}\right)\right)\left(w - \wp\left(\frac{\omega_2}{2}\right)\right)\left(w - \wp\left(\frac{\omega_1+\omega_2}{2}\right)\right)}}} = z - z_0$$

(check this by differentiating w.r.t. z) As usual, need to choose signs in the square root function to match $\wp'(z)$.

And choose from z_0 to z avoiding poles of $\wp'(z)$, integrate over its image under $\wp$.

Thus as the composition

$$z \longmapsto \wp(z)$$
$$\xi \longmapsto \int_{\xi_0}^{\xi} \frac{dw}{\sqrt{(-)}}$$

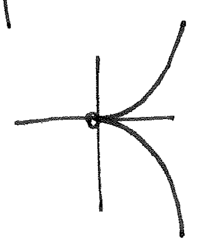
is equal to z (up to const.) then $\wp(z)$ is an inverse to the elliptic integral.

Aside: elliptic curve: genus one proj. variety of dimension 1.

More concretely, set of cx. points (x,y) $x,y \in \mathbb{C}$ satisfying the equation $y^2 = x^3 + Ax + B$

with $\Delta(E) = -16(4A^3 + 27B^2) \neq 0$ ← ensures curve is not "singular" – has well defined tangent plane

Graph real points of $y^2 = x^3$ (so (x,y) both real)



cusp at origin.

Δ has simpler expression in terms of differences of roots:

$$E: \quad y^2 = p(x) \quad \text{has} \quad \Delta(E) = \underbrace{c_1^2}_{\text{leading coeff.}} \cdot \prod_{i<j} (r_i - r_j)^2 \qquad r_i: \text{roots of } p.$$

by our earlier result,

the ~~elliptic curve~~ equation $\quad y^2 = 4x^3 - 60\,G_4\,x - 140\,G_6$

defines a (non-singular) elliptic curve.

<u>Thm</u>:

$$\begin{aligned} \phi : \mathbb{C}/M &\longrightarrow E \subset \mathbb{P}^2(\mathbb{C}) \\ z &\longmapsto [\wp(z), \wp'(z), 1] \end{aligned}$$

is a cx. analytic isomorphism of cx. Lie gps.

<u>pf</u>: Silverman, Ch. VI Prop. 3.6(b) "Arithmetic of Elliptic Curves"

We have shown that $\mathrm{Im}(\phi)$ satisfies equation for E.