Now define analogous constructions on Riemann surface X to those on $\mathbb{C}$:

whenever these are local considerations, we can just use charts.

E.g. Say f is holomorphic at $p \in X$ (f: cx-valued function on open set W of X)

if $\exists$ chart $\phi : \mathcal{U} \to V \subseteq \mathbb{C}$

s.t. $f \circ \phi^{-1}$ is holomorphic.

(Note this is independent of choice of chart, since if $f \circ \phi_1^{-1}$ holomorphic

then $f \circ \phi_2^{-1} = \underbrace{f \circ \phi_1^{-1}}_{\text{holom.}} \circ \underbrace{\phi_1 \circ \phi_2^{-1}}_{\text{holom. since it is transition function}}$ is holomorphic.)

so any chart $\phi : \mathcal{U} \to V \subseteq \mathbb{C}$ is itself a holomorphic function

$(\phi \circ \phi^{-1}(z) = z \ \checkmark)$

also check that sums / products / quotients of holom. functions on X (with denom. $\neq 0$) are holomorphic

When applied to affine plane curve, then if $X = \{(z,w) \mid f(z,w) = 0\}$

then π_z, π_w are charts so holomorphic. $\Rightarrow$ so is any polynomial in π_z, π_w. expressed as poly. $g(z,w)$. restricted to X.

play similar games with projective plane curves using homogeneous ~~[illegible]~~ polys.

Miranda's notation: $\mathcal{O}_X(W)$: holomorphic functions on open set $W \subseteq X$.

Similarly define isolated singularities (and classify them): cx-valued function f has isolated singularity at p if $\exists$ nbhd $\mathcal{U}$ of p, chart $\phi : \mathcal{U} \to \mathbb{C}$ s.t. $f \circ \phi^{-1}$ has isolated singularity.

Example of <u>meromorphic</u> function on X:

$X = \mathbb{P}^1 := \{[z_1 : z_2]\}$ $\qquad$ p, q homog. polynomials of <u>same</u> deg. d

then $p(z_1, z_2)/q(z_1, z_2)$ $(q \not\equiv 0)$ defines a meromorphic function.

What about polynomials $p(z_1, z_2)$? Are they holomorphic on all of X (i.e. in $\mathcal{O}(X)$)?

No, since have poles at ∞, unless constant.

(check f : function on nbhd of ∞ in S^2 with coord z is holomorphic $\Longleftrightarrow$ $f(1/z)$ is holomorphic at $z = 0$.)

Are there any non-const. holom. functions on $\mathbb{P}^1$? On $\mathbb{C}/M$, there weren't.

need additional machinery – tools from cx. analysis.

① Laurent expansions.

f : holom on punctured nbhd of p so $\exists$ $\phi : \underset{p}{\overset{\cup\!\!\!\shortmid}{\mathcal{U}}} \to V$ s.t. $f(\phi^{-1}(z))$ holomorphic in nbhd of $\phi(p) =: z_0$

Then have Laurent expansion for $f \circ \phi^{-1}$ about z_0:

$$f(\phi^{-1}(z)) = \sum_n c_n (z - z_0)^n$$

Issue: series depends on choice of chart ϕ.

What can we hope to preserve among all choices? order.

Define $\operatorname{ord}_p(f) = \min\{c_n \mid c_n \neq 0\}$

<u>Lemma</u>: $\operatorname{ord}_p(f)$ is independent of chart choice.

Pf: Given $p \in \mathcal{U}_1 \cap \mathcal{U}_2$ $\qquad$ $\phi_1 : \mathcal{U}_1 \to V_1$ $\quad$ $\phi_1(p) = z_0$ local coord. z

$\phi_2 : \mathcal{U}_2 \to V_2$ $\quad$ $\phi_2(p) = w_0$ local coord w

$T(w) = \phi_1 \circ \phi_2^{-1} =: \underset{\sim}{z}$ on $\phi_2(\mathcal{U}_1 \cap \mathcal{U}_2)$. $\quad (w_0 \mapsto z_0)$

local coord. on $V_1 = \phi_1(\mathcal{U}_1)$

So can write z in a nbhd of z_0 as powers series

$$z = T(w) = T(w_0) + \sum_{n \geq 1} a_n (w - w_0)^n$$

$$= z_0 + \sum_{n \geq 1} a_n (w - w_0)^n$$

note: $a_1 \neq 0$ else $T'(w_0) = 0$ (can't happen with transition fns, which are invertible)

So if we start with Laurent series in z for f:

$f \circ \phi_1^{-1} = c_{n_0} (z - z_0)^{n_0} +$ higher order terms $(c_{n_0} \neq 0.)$

get series in w by composing with above series : $f \circ \phi_1^{-1} \circ (\phi_1 \circ \phi_2^{-1})$

Get $f \circ \phi_2^{-1} = c_{n_0} a_1^{n_0} (w - w_0)^{n_0} +$ higher order terms

Since $c_{n_0} \neq 0$ by assumption, $a_1 \neq 0$ by above, order is preserved.

Now easy to check : ① if f, g holomorphic (or meromorphic) so is $f \pm g$, $f \cdot g$, (f/g provided $g \neq 0$ (or $g \not\equiv 0$), respectively)

never 0 on open set.

not identically 0

② $\operatorname{ord}_p(fg) = \operatorname{ord}_p(f) + \operatorname{ord}_p(g)$

$\operatorname{ord}_p(f/g) = \operatorname{ord}_p(f) - \operatorname{ord}_p(g)$

$\operatorname{ord}_p(f \pm g) \geq \min\{\operatorname{ord}_p(f), \operatorname{ord}_p(g)\}$ ← Taylor/Laurent series coeffs may cancel when adding/subtracting.

complex chart is homeomorphism, bijection ~~[illegible]~~ so many thms from cx. analysis immediate:

<u>Thm</u>: Zeros of holomorphic function are isolated.
(on open set of X = Riemann surface)

<u>pf</u>: zeros of f are zeros of $f \circ \phi^{-1}$, ϕ: chart s.t. $f \circ \phi^{-1}$ holom.

<u>Corollary</u>: f, g meromorphic on open set, $f = g$ on set with a limit point. Then $f = g$ on the open set.

<u>pf</u>: $f - g$ either 0 or has isolated zeros.

Maximum <u>modulus</u> theorem: f holomorphic on open, connected set $\Omega \subseteq X$, $p \in \Omega$ s.t. $|f(x)| \le |f(p)|$ $\forall x \in \Omega$ $\Rightarrow$ f constant.

<u>pf</u>: Max. mod. for $f \circ \phi^{-1}$ with max $\phi(p)$. $\Rightarrow$ $f \circ \phi^{-1}$ constant $\Rightarrow$ f constant since ϕ^{-1} homeom.

<u>Corollary</u>: X compact Riemann surface, f holom. on all of X. Then f is constant

<u>pf</u>: $|f|$ achieves maximum on X since compact. Max. mod principle $\Rightarrow$ f constant.
(since X open in X connected.)

(Nice comment in Miranda about Liouville's thm relating to this on $X = S^2$)

Further <u>results</u>: Can define f to be C^∞ or harmonic at $p \in X$ if $f \circ \phi^{-1}$ for some chart $\phi : \underset{p}{\overset{\cup}{U}} \to \mathbb{C}$

check that harmonicity is independent of chart.
(need to use that transition maps are holomorphic so real/imag parts are harmonic, use C-R equations in taking derivs via chain rule.)