On Wednesday,

discussed two new properties of analytic maps:

f analytic, $f'(z_0) \neq 0$ then f conformal at z_0 (angle-preserving, scales uniformly in all directions)

various converses are true:

assume both properties or

just one of them + sufficient regularity (f has continuous first partials)

f analytic, $f = u + iv$, then u, v harmonic $\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \right.$, same for $\left. v \right)$

Proposition: u, v harmonic conjugates, then if $u = c_1$, $v = c_2$ smooth curves,

then u, v intersect orthogonally (see p. 4 of Wed. notes)

Second pf uses cx. inverse function thm:

$f: \Omega \to \mathbb{C}$ analytic, ~~...~~ $f'(z_0) \neq 0$, then

$\exists$ nbhd $\mathcal{U}$ of z_0, V of $f(z_0)$ s.t. $f: \mathcal{U} \to V$ is a bijection.

Then f has well-defined inverse f^{-1} on V which is

analytic with derivative $\frac{d}{dw}(f^{-1})(w) = \frac{1}{f'(z)}$

where $w = f(z)$.

(application of real-variable inverse function thm, given by Jacobian condition, + C-R eq'ns.)

pf: $u(x,y) = c_1$ smooth if $\text{grad}(u) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \neq 0$ for (x,y) with $u(x,y) = c_1$

(and grad(u) is perpendicular to tangent vector at u.)

just chain rule applied to

$$\underbrace{\frac{d}{dt}\left(u(x(t), y(t)\right)}_{\text{grad}(u) \cdot (x'(t), y'(t))} = \frac{d}{dt}(c_1) = 0$$

show that normal vectors of u, v perpendicular.

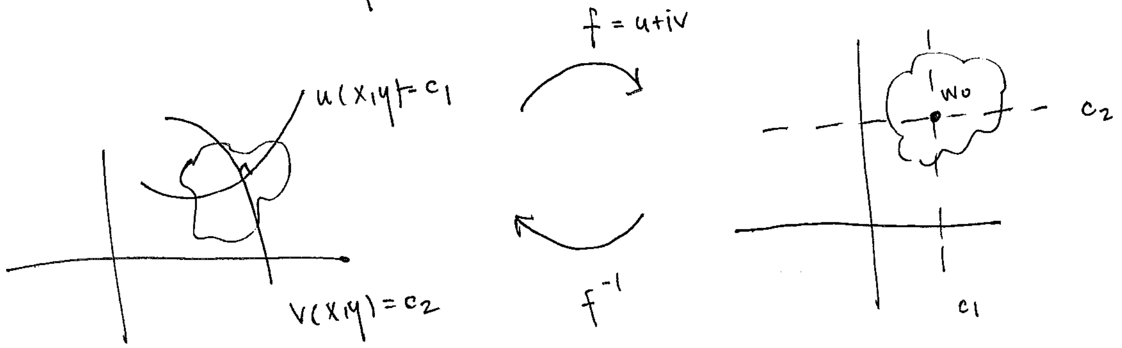
i.e. compute $\text{grad}(u) \cdot \text{grad}(v) = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} = 0$

by C-R eqns.

pf 2: f analytic, $f'(z_0) \neq 0$, then

f^{-1} analytic in nbhd. of $w_0 = f(z_0)$ with $f^{-1\prime}(w_0) \neq 0$ by inverse function thm., hence conformal at w_0.

ex inv. function thm ~~...~~ real-var follows from inv. fn. thm + Cauchy-Riemann eqns.



But $u(x,y) = c_1$ is image of line c_1 under f^{-1}

Converse? Given u harmonic, does there exist v s.t.

$u + iv$ is analytic?

Yes, at least in a nbhd. of any point z_0.

Just solve C-R eqns. Example: $u = x^2 - y^2$. Find v.

(Know answer from example last time, but...)

C-R:

$$\underbrace{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}_{} = 2y \qquad \frac{\partial v}{\partial y} = \underbrace{\frac{\partial u}{\partial x}}_{} = 2x$$

Solve: $2yx + \phi(y)$ Solve: $2xy + \psi(x)$

So $\phi(y) = \psi(x)$ must be constant.

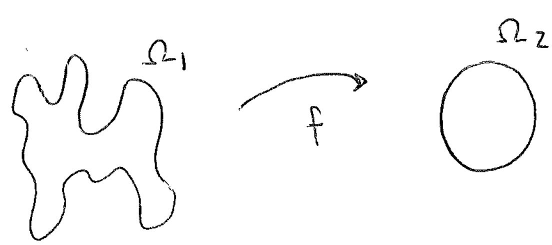
Replace $2y$, $2x$ by general $-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}$ and solve: in nbhd of $z_0 = x_0 + iy_0$.

$$v(x_1, y_1) = \int_{y_0}^{y_1} \frac{\partial u}{\partial x}(x_1, y)\,dy - \int_{x_0}^{x_1} \frac{\partial u}{\partial y}(x, y_0)\,dx + \underbrace{c}_{\text{Some const.}}$$

$= v(x_0, y_0)$

Plan: Solve problem about harmonic functions on a complicated region Ω_1, map conformally, bijectively to Ω_2

Solve problem on Ω_2, map back.



To Validate this strategy, we have a pair of easy propositions.

Stronger converse: Let Ω be a region, u harmonic on Ω
(nec. twice diff.)

then u is smooth (has derivatives of all orders) and in nbhd. of any $z_0 \in \Omega$,

u is the real part of an analytic function. If Ω simply connected,

then $\exists$ f analytic with $u = Re(f)$ on all of Ω.

pf: Given u harmonic, easy way to construct analytic function:

$$g := \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}$$

(Cauchy Riemann
$\frac{\partial(Re)}{\partial x} = \frac{\partial(Im)}{\partial y}$ since u harmonic
$\frac{\partial(Re)}{\partial y} = -\frac{\partial(Im)}{\partial x}$ since mixed partials equal)

If Ω simply conn., $\exists$ analytic f s.t.

$f' = g$ on Ω.

(p. 107 of Ahlfors: $\int_\gamma f\,dz = 0$ γ closed $\Rightarrow \exists F$ s.t. $F' = f$)

the assumption is true by Cauchy's thm if Ω simply conn.

$\Rightarrow$ $u = Re(f)$ (take derivs of $f = u_1 + iv_1$.

$$f' = \frac{\partial u_1}{\partial x} + i \frac{\partial v_1}{\partial x} = \frac{\partial u_1}{\partial x} - i \frac{\partial u_1}{\partial y}$$

so $u_1 = u + c$.)

change f by c accordingly.

Other statements follow easily from this.

Proposition : (i) $f : \Omega_1 \to \Omega_2$ conformal, bijective, then

$f^{-1} : \Omega_2 \to \Omega_1$ is conformal.

(ii) $f : \Omega_1 \to \Omega_2$, $g : \Omega_2 \to \Omega_3$ conformal. bijective, then $g \circ f$ is confomal, bijective.

pf : immediate. (i) f^{-1} exists since f bijective. Inverse function thm $\Rightarrow$

f^{-1} analytic with $\frac{d}{dw}(f^{-1}) = 1 / \frac{d}{dz}(f)$

with $f(z) = w$.

(so in particular $\frac{d}{dw}(f^{-1}) \neq 0$ so f^{-1} conformal)

(and defined when f conformal)

(ii) compositions of analytic bijections are analytic bijections. chain rule gives non-zero deriv.

(so in particular, bijective conformal maps of Ω to itself forms gp.)

Proposition 2 : u harmonic on Ω_2, $f : \Omega_1 \to \Omega_2$ analytic, then $u \circ f$ harmonic on Ω_1

pf : Pick $z \in \Omega_1$ $w = f(z)$. $\mathcal{U}$: nbhd of w in Ω_2

$V := f^{-1}(\mathcal{U})$. : nbhd. of z.

Want to show $u \circ f$ harmonic on V. (suffices, since being harmonic is local condition)

Here we use that u : harmonic $\Rightarrow$ $\exists\ g$ on $\mathcal{U}$ s.t. $u = \mathrm{Re}(g)$.

Then $u \circ f = \mathrm{Re}(g \circ f)$ but $g \circ f$ analytic so $\mathrm{Re}(g \circ f)$ harmonic.