Maps between Riemann surfaces : X, Y pair of Riemann surfaces

$F : X \to Y$ is <u>holomorphic</u> at $p \in X$ iff there

exists charts $\phi_X : \mathcal{U}_X \to V_X$ $\phi_Y : \mathcal{U}_Y \to V_Y$ with

$p \in \mathcal{U}_X$, $F(p) \in \mathcal{U}_Y$ s.t. $\phi_Y \circ F \circ \phi_X^{-1} : V_X \subseteq \mathbb{C} \to V_Y \subseteq \mathbb{C}$

is holomorphic at $\phi_X(p)$. Similarly F is holomorphic on open set $\Omega \subseteq X$ if

holomorphic at every point $p \in \Omega$.

first <u>examples</u> : ① $id : X \to X$ is holomorphic (on all X) since

charts are holomorphic functions (reserve term "map" for $X \to Y$
"function" as short-hand for
" cx.-valued function" $X \to \mathbb{C}$)

② if $Y = \mathbb{C}$, then $F : X \to \mathbb{C}$
is holomorphic map if F is holomorphic (cx-valued) function.

(still check this latter condition using charts on X)

Easy <u>lemmas</u> : Ⓐ checking holomorphicity at p for one chart ("there exists")
is same as checking for all charts. (transition functions are holomorphic)

Ⓑ Can analogously define $F : X \to Y$ continuous or C^∞ (i.e. smooth)
then analytic $\Rightarrow$ continuous & C^∞.

Ⓒ Composition of holom. maps is holomorphic map.

Ⓓ Composition of holom map + holom. function $X \xrightarrow{F} Y \xrightarrow{f} \mathbb{C}$
is holom. function on $F^{-1}(Y)$. More generally on $F^{-1}(\text{domain}(f))$

Ⓔ Composition of holom. map F + merom. function f is merom. function on $F^{-1}(\text{domain}(f))$
provided the image$(F) \not\subseteq$ poles(f).

Fancy language: Riemann surfaces and holomorphic maps form category.

(objects: surfaces morphisms: holomorphic maps, identity, associativity)

Given $F: X \to Y$ we have contravariant functor giving rise to

$\mathbb{C}$-algebra homomorphism $F^* : \underbrace{\mathcal{O}_Y(W)}_{\substack{\text{holom. functions} \\ \text{on open set } W \subseteq Y}} \longrightarrow \mathcal{O}_X(F^{-1}(W))$

(similarly have F^* on meromorphic functions $\mathcal{M}_Y(W) \to \mathcal{M}_X(F^{-1}(W))$.)

$g \longmapsto F^*(g) = g \circ F$

Easy to verify the relation : $F^* \circ G^* = (G \circ F)^*$

category theory really only useful to us in reminding us to choose morphisms compatible with structure of objects (e.g. holomorphic maps for Riemann surfaces)

and by providing basic constructions of maps to draw on.

e.g. isomorphism in category: morphism and "inverse" morphism whose composition is identity.

(necessarily bijection)

So for us, isomorphism of Riemann surfaces is holomorphic map with holomorphic inverse (necessarily bijection).

Check that $\mathbb{P}^1 \longrightarrow C_\infty \subseteq \mathbb{R}^3$ defines isom. of Riemann surfaces

$$[z:w] \longmapsto \left(2\,\mathrm{Re}(z\bar{w}),\ 2\,\mathrm{Im}(z\bar{w}),\ |z|^2-|w|^2\right)\cdot\frac{1}{|z|^2+|w|^2}$$

(checked in h.w. this is homeomorphism)

This map was concocted using inverse of chart map obtained from stereographic projection.

e.g. chart on $\mathbb{P}^1$: $[z:w] \mapsto z/w$

$\varphi^{-1}: [x:1] \leftarrow\!\shortmid\ x$

chart of C_∞: e.g. proj. from north pole

$$\phi_1(x,y,w) = \frac{x}{1-w} + i\,\frac{y}{1-w}$$

$$\phi \circ F \circ \psi^{-1}(x) = \phi \circ F([x:1]) = \phi\left(\frac{2\,\mathrm{Re}(x)}{|x|^2+1}, \frac{2\,\mathrm{Im}(x)}{|x|^2+1}, \frac{|x^2|-1}{|x|^2+1} \right)$$

$$= \frac{2 \cdot \mathrm{Re}(x) + i\, 2\,\mathrm{Im}(x)}{2} = x. \quad \checkmark$$

Easy theorems about holomorphic maps (follow directly from analogues for holomorphic functions)

I: Open Mapping Thm: $F: X \to Y$ non-const holom. map, then F is open.

II: $F: X \to Y$ holom. map, one-one, then $F: X \to F(X)$ is isomorphism.
(apply inverse function thm for holom. functions)

III: F, G holom. maps $: X \to Y$ s.t. $F = G$ on set with limit point in X, then $F = G$.

Preimages: First a lemma: X compact, $F: X \to Y$ non-const holom. map then Y is compact and F onto.

pf: F holom, X open as total space, so $F(X)$ open by open mapping thm (requires F non-const.)
X compact $\Rightarrow$ $F(X)$ compact $\Rightarrow$ $F(X)$ closed (since Y Hausdorff)
Thus $F(X)$ is open, closed in Y, and Y connected so $F(X) = Y$ showing F onto and Y compact

Proposition: $F: X \to Y$ non-const map. For each $y \in Y$, $F^{-1}(y)$ is discrete in X.
In particular, if X, Y compact, $F^{-1}(y)$ is (non-empty) finite set.

pf: Note second statement follows immediately from the first by Lemma. Now to prove first... Given $y \in Y$ pick charts and local coords for y, and $x \in F^{-1}(y)$
Call them z, w respectively so that $\phi \circ F \circ \psi^{-1}$ is non-const holom. function $z = g(w)$ some g. Can choose charts so that $y \mapsto 0 \in \mathbb{C}$
$x \mapsto 0 \in \mathbb{C}$

Thus g has $g(0) = 0$. Since zeros of holom. function are isolated,

then x must have a nbhd. in X with no other zeros (i.e. no other preimages of $y \in Y$)

$\Rightarrow$ $F^{-1}(y)$ is discrete. //

Final observation. $Y = \mathbb{C}$ then holom. maps from $X \to Y$ are same as holomorphic functions. Similarly, if f : meromorphic function on X

can define map $X \longrightarrow \mathbb{C}_\infty$: Riemann sphere

$$F : x \longmapsto \begin{cases} f(x) & \text{if } x \text{ not a pole} \\ \infty & \text{if } x \text{ is a pole} \end{cases}$$

∞ realized as north pole in our embedding in $\mathbb{R}^3$.

<u>check</u> : resulting F is holomorphic map, and induces 1-1 correspondence

$$\left\{ \begin{array}{c} \text{meromorphic functions} \\ \text{on } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{holomorphic maps} \\ F : X \to \mathbb{C}_\infty \end{array} \right\}$$

(except for const. function $x \mapsto \infty$ $\forall x \in X$)

Next we explore remarkable local structure of holomorphic maps.

When written as holomorphic function in local coords, just power map $z \mapsto z^n$ for some n.