If Y has "reasonable" local structure (for us, with Riemann surfaces our local structure is disc!)

e.g. locally path connected

and $\underbrace{\text{semi-locally}}$ simply conn'd.

loops in nbhd $\mathcal{U}$ are contractible.

but contractions can occur in Y.

(not just restricted to $\mathcal{U}$)

then there is a 1-1 correspondence between:

- equivalence classes of coverings $F: X \to Y$, X: connected.

(so data here is pair $(X, F: X \to Y)$)

- conjugacy classes of subgps of $\pi_1(Y, y_0)$.

[Recall, two coverings $F: X \to Y$, $F': X' \to Y$ are equivalent if $\exists$ homeomorphism $\phi: X \to X'$ such that $F = F' \circ \phi$]. (See Section 1.3 in Hatcher)

Correspondence is given as follows: Given $F: X \to Y$, there is an induced homomorphism $F_*: \pi_1(X, x_0) \to \pi_1(Y, F(x_0))$

$$[\gamma] \longmapsto [F \circ \gamma]$$

for $\pi_1(Y, y_0)$ pick $x_0 \in F^{-1}(y_0)$ and then subgp. for $F: X \to Y$ is just image $F_*(\pi_1(X, x_0))$. Different choices of $x_0 \in F^{-1}(y_0)$ give different conjugates of this subgp in $\pi_1(Y)$.

Now want to show that a covering can be constructed for any subgp. of $\pi_1(Y)$.

E.g. trivial subgp. $G: \tilde{Y} \to Y$ with $\pi_1(\tilde{Y})$ trivial.

"universal cover"

$$\tilde{Y} = \{(y, A) \mid y \in Y,\ A: \text{homotopy class of paths in } Y \text{ from } y_0 \text{ to } y\}$$

$G((y, A)) = y$. Need to check that give $\tilde{Y}$ topology so that G is covering map.

Called universal cover b/c satisfies universal property: If $F: X \to Y$ is any connected covering space then $\exists!$ covering map $F': \tilde{Y} \to X$ s.t. $G = F \circ F'$.

—

Have action of $\pi_1(Y, y_0)$ on $\tilde{Y}$, given by concatenating a path representing A in (y, A) with a loop based at y_0.

Then $Y = \tilde{Y} / \pi_1(Y, y_0)$, and for any subgp. $H \subset \pi_1(Y, y_0)$ define the covering space $Y_H = \tilde{Y}/H$.

—

Basic ingredient in pfs is notion of path lifting:

$F: X \to Y$ any map. $\gamma: [0,1] \to Y$ a path

Given $x_0 \in Y$ with $F(x_0) = \gamma(0)$, a lift of γ is a path $\tilde{\gamma}: [0,1] \to X$ with $F \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = x_0$.

<u>Proposition</u> ① If $F: X \to Y$ local homeom., then path lift (with given initial point) is unique, if it exists. If $F: X \to Y$ is covering map, then such a path lift always exists.

② If $F: X \to Y$ local homeom, γ_0, γ_1 paths in Y with same endpts and are homotopic (through maps w/ fixed endpts) through paths <u>with lifts</u> $\tilde{\gamma}_s$ whose initial pt is all same $x \in X$. then $\tilde{\gamma}_s(1) = \tilde{\gamma}_0(1)$ for all $s \in [0,1]$.

Example: $X = \mathbb{C}/M$. Natural projection map $\pi : \mathbb{C} \to X$

which is a covering map (onto map with $\pi^{-1}(U_x)$ disjoint open sets of $\mathbb{C}$ which are homeom. onto U_x)

Also $\pi_1(\mathbb{C}) = \{1\}$, so $\mathbb{C}$ must be universal cover of X.

said that $\tilde{Y} / \pi_1(Y) \simeq Y$ with action given lifts of loops generating $\pi_1(Y)$.

$\pi_1(X)$ = free gp. on two generators $\simeq M$, so action on $\mathbb{C}$ is just translation by M.

—

When are $X = \mathbb{C}/L$, $Y = \mathbb{C}/M$ isomorphic? M, L lattices in $\mathbb{C}$.

$F : X \to Y$ holomorphic map. for convenience, $F(0) = 0$ by composing by biholomorphic map $y \mapsto y + q$ for some $q \in Y$.

Hurwitz' formula:

$$-\underbrace{\chi(X)}_{0} = -\underbrace{\chi(Y)}_{0} + \text{error from ramification}$$

since genus $g = 1$ $\Rightarrow$ so no ramification.

say "F is unramified"

That means that F is covering map (onto since X compact, F holom. inverse image of open set is disjoint union of opens $= \deg(F)$)

$\pi : \mathbb{C} \to X$ is covering map

so have diagram:

$$\begin{array}{ccc} \mathbb{C} & \overset{\text{homeom } G}{\dashrightarrow} & \mathbb{C} \\ {\scriptstyle \pi_L}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ X & \underset{F}{\longrightarrow} & Y \end{array}$$

where all maps covering maps, + holomorphic maps.

By uniqueness of univ. cover, $\exists$ homeom. G making diagram commute.

might as well take $G(0) = 0$ since translation in M doesn't affect π.

and this G must be holomorphic covering map with $G(0) \in M$: lattice of Y

Commutativity of diagram implies $G(z+\ell) \equiv G(z) \mod M$ for all $\ell \in L$, $z \in \mathbb{C}$

i.e. $G(z+\ell) - G(z) =$ lattice point dep. on z, ℓ, call it $\omega(z,\ell)$

Since M discrete, then for fixed ℓ, $\omega(z,\ell)$ indep. of z.

$\Rightarrow$ $G'(z+\ell) = G'(z)$ so G' invariant under translations in L.

$\Rightarrow$ G' bounded (all values in range occur in fundamental //-ogram)

$\Rightarrow$ G' constant $\Rightarrow$ G linear (set $G(0)=0$ so $G(z) = \gamma z$ for some γ)

Note $G(z+\ell) \equiv G(z) \mod M \Rightarrow G(\ell) \equiv G(0) = 0 \mod M$

i.e. $G(L) \subseteq M$. so $\gamma \cdot L \subseteq M$ with γ as above.

$\Rightarrow$ F is a group homom.

(it maps $0 \to 0$. G is a group homom. π_L, π_M are group homoms.

Given $a, b \in X$ then lift to $a', b' \in \mathbb{C}$ s.t. $\pi_L(a'), \pi_L(b') = a, b$.

then $G(a'+b') = G(a') + G(b')$

then $\pi_M(G(a') + G(b')) = \pi_M(G(a')) + \pi_M(G(b'))$

$= F(a) + F(b)$

by commutativity of diagram.

This was indep. of lift of $a, b \in X$

since $G(L) \subseteq M$.

We have proved (Prop. 1.11 in Ch. 3 of Miranda)

Any holomorphic map $F : X \to Y$ is induced by a linear map

$G : \mathbb{C} \to \mathbb{C} \quad : \quad \gamma z + a \quad$ with $\quad \gamma(L) \subseteq M, \quad a = 0 \iff F(0) = 0.$

H is an isomorphism of groups iff $\gamma L = M$ and in general, $\deg(F) = |M / \gamma L|$

Corollary: The group of automorphisms fixing 0 of X is either:

$\mathbb{Z}/2\mathbb{Z}$ (with L: not square or hex.)

$\mathbb{Z}/4\mathbb{Z}$ (with L: square)

$\mathbb{Z}/6\mathbb{Z}$ (with L: hexagonal)

pf: if $\begin{array}{l} F : X \to X \\ 0 \mapsto 0 \end{array}$ then it is induced by map $G : \begin{array}{l} z \mapsto \gamma z \\ \mathbb{C} \to \mathbb{C} \end{array}$

with $\gamma L = L$

this requires $\|\gamma\| = 1$. $\gamma = \pm 1$ possibilities. Suppose γ not real.

If ℓ is elt. of L of minimal length, then so is $\gamma \ell$.

these are independent over $\mathbb{R}$ so $\langle \ell, \gamma \ell \rangle = L$.

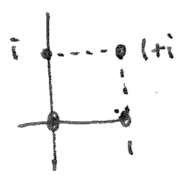
So is $\gamma^2 \ell$. (of min length) Hence $\gamma^2 \ell = m \gamma \ell + n \ell$

so $\gamma^2 - m\gamma + n = 0$

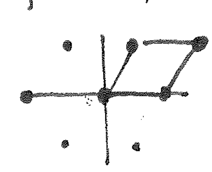
$\to$ γ is small root of unity.

(either i, $e^{(2\pi i/3)}$ or some power of them)

i: square lattice



$e^{2\pi i/3}$: hex lattice



A similar analysis for two different lattices : $L_{\langle 1,\tau\rangle}$, $L_{\langle 1,\tau'\rangle}$

shows $\gamma L_{\langle 1,\tau\rangle} = L_{\langle 1,\tau'\rangle} \iff \exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

with $\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau' \\ 1 \end{pmatrix}$

(Here $\tau \in \mathcal{H}$: upper half plane)

$$L_{\langle \omega_1 ; \omega_2 \rangle} \xmapsto{\sim} L_{\langle 1,\tau\rangle}$$

$$z \longmapsto \frac{1}{\omega_1} z$$