All definitions are otherwise same. In particular, if S compact then $\Omega^i_{S,c} = \Omega^i_S$. (If S non-compact, then we have $H^0_c(S) = 0$ since constants aren't compactly supported.)

Then $\Omega^2_{S,c} \to \mathbb{R}$ is well-defined since $\mathrm{supp}(\rho)$ compact.

$\rho \mapsto \int_S \rho$

claim: If S connected, oriented, smooth (not nec. compact), this ~~is~~ induces isomorphism $H^2_c(S) \simeq \mathbb{R}$.

(again clear that if $\rho = d\alpha$, then $\int_S \rho = 0$ by Stokes thm.)

$= \int_{\partial S} \alpha$ with $\partial S = \emptyset$.

pf: show true for $\mathbb{R}^2$, then use partition of unity to extend to general case of ρ on S with supp in compact K covered by open sets $U_1, \dots, U_n$ having structure of $\mathbb{R}^2$.

For $\mathbb{R}^2$, write $\rho = R(x_1, x_2)\, dx_1 dx_2$

(surjective: choose smooth, comp. supp. R with non-zero integral.)

injective: similar flavor to our proof on torus. Suppose $\int_S \rho = 0$.

pick ψ on $\mathbb{R}$ with compact support, total integral $\int_{-\infty}^{\infty} \psi(t)\,dt = 1$.

Set $r(x_1) = \int_{-\infty}^{\infty} R(x_1, t)\,dt$ (well-defined since R comp. supp.)

and $\tilde{R}(x_1, x_2) := R(x_1, x_2) - r(x_1)\psi(x_2)$ another comp. supp. function.

By construction, $\displaystyle\int_{-\infty}^{\infty} \tilde{R}(x_1,t)\,dt = 0$

$$\left(= \int_{-\infty}^{\infty} \left(R(x_1,t) - r(x_1)\psi(t) \right) dt = \int_{-\infty}^{\infty} R(x_1,t)\,dt - r(x_1) \checkmark. \right)$$

Define $\displaystyle P(x_1,x_2) = \int_{-\infty}^{x_2} \tilde{R}(x_1,t)\,dt$. It has compact support, since if $\tilde{R}(x_1,t)$, as function of t, is supported for $t \in [a,b]$ then if $x_2 < a$, $P = 0$. Since $\displaystyle\int_{-\infty}^{\infty} \tilde{R}\,dt = 0$, this implies $P = 0$ if $x_2 > b$ since $\displaystyle\int_{-\infty}^{x_2} = -\int_{x_2}^{\infty}$

Of course, $\dfrac{\partial P}{\partial x_2} = \tilde{R}$.

Set $\displaystyle Q(x_1,x_2) = \psi(x_2)\int_{-\infty}^{x_1} r(t)\,dt$

$$= \psi(x_2)\int_{-\infty}^{x_1}\int_{-\infty}^{\infty} R(t_1,t_2)\,dt_2\,dt_1$$

Since $\displaystyle\int_{\mathbb{R}^2} R\,dx_1\,dx_2 = 0$, then Q has compact support (similar to ↗)

and $\dfrac{\partial Q}{\partial x_1} = \psi(x_2)\,r(x_1)$ by def'n

so $\alpha = -P\,dx_1 + Q\,dx_2$ satisfies $d\alpha = \rho$, i.e. ρ exact!

Calculus on Riemann surfaces - so far, we've studied forms/cohomology for smooth oriented surfaces, and now we want to add cx. structure.

Not so hard: cotangent space at $p \in X$: $T^*X_p = \mathrm{Hom}_{\mathbb{R}}(TX_p, \mathbb{R})$

and to any smooth function on X, call it f, we associate an element of T^*X_p by computing df

$$df = \left.\frac{\partial f}{\partial x_1}\right|_p dx_1 + \left.\frac{\partial f}{\partial x_2}\right|_p dx_2 .$$

now we "complexify": $T^*X_p^{\mathbb{C}} = \mathrm{Hom}_{\mathbb{R}}(TX_p, \mathbb{C})$

and put complex structure on TX_p. Do this explicitly in a moment, and in a coordinate free way.

but want to first note that we can do it

"complex structure" on real vector space V is $\mathbb{R}$-linear map $J: V \to V$ such that $J^2 = -1$.

Easy fact: $\exists!$ cx. structure on TX_p such that the derivative of a holomorphic function at p is complex linear.

Recall a map $A: V \to \mathbb{C}$ is cx. linear if $A(Jv) = iAv \ \forall v \in V$.

On first day of last semester, defined cx. numbers explained various incarnations.

One was matrices of form $\begin{pmatrix} \alpha & -\beta \\ +\beta & \alpha \end{pmatrix} \simeq \mathbb{C} = \{\alpha + i\beta\}$.

Say, similarly, that A is anti-linear $: V \to \mathbb{C}$ if

$$A(Jv) = -i\,A(v) \qquad \forall\ v \in V.$$

Another easy fact: Any $\mathbb{R}$-linear map may be decomposed into in a unique way sum of cx. linear and anti-linear maps:

$$A = A' + A'' \qquad A' : \text{linear} \quad \tfrac{1}{2}\left[A(v) - i\,A(Jv)\right]$$

$$A'' : \text{anti-linear} \quad \tfrac{1}{2}\left[A(v) + i\,A(Jv)\right]$$

so can decompose cotangent space at p: $T^*X_p^{\mathbb{C}} = T^*X_p' \oplus T^*X_p''$

and have corresponding decomp of $\Omega^1_{X,\mathbb{C}} = \underbrace{\Omega^{1,0}_X}_{\text{cx. linear}} \oplus \underbrace{\Omega^{0,1}_X}_{\text{cx. anti-linear}}$

where $\Omega^{1,0}_X$ means element is in T^*X_p' $\forall\ p \in X$. According to our first easy fact, if f holomorphic then $df \in \Omega^{1,0}_X$ while $\bar{f}$ has $df \in \Omega^{0,1}_X$.

Have maps:

$$\begin{tikzcd}
\Omega^0_X \arrow[r] \arrow[d] & \Omega^{0,1}_X \arrow[d] \\
\Omega^{1,0}_X \arrow[r] & \Omega^2_X
\end{tikzcd}$$

Write these explicitly using coordinates.

$dz = dx + i\,dy$ is basis for T^*X_p' if x, y local coords at p.

$$J : \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} -dy \\ +dx \end{pmatrix}$$

and $d\bar{z} = dx - i\,dy$ is anti-linear.

mult. by i: $dx + i\,dy \mapsto i\,dx - dy$

so locally, a $(1,0)$ form is

$\alpha_1\, dz \qquad \alpha_1$: function on X

$(0,1)$ form is $\alpha_2\, d\bar{z} \qquad \alpha_2$: function on X.

Then $dx = \frac{1}{2}(dz + d\bar{z})$

$dy = \frac{1}{2i}(dz - d\bar{z})$

Then $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \underbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)}_{\text{call this } \frac{\partial f}{\partial z}} dz + \underbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)}_{\text{call this } \frac{\partial f}{\partial \bar z}} d\bar z$$

as expected, where

$\frac{\partial f}{\partial \bar z} = 0$ is just Cauchy-Riemann equations.

or sometimes even more succinctly $\partial, \bar\partial$ resp.

so if f holomorphic, then $df = f'(z)\,dz$

If we understand $\partial(A\,d\bar z) = \frac{\partial A}{\partial z} dz \wedge d\bar z$

$$\bar\partial(B\,dz) = \frac{\partial B}{\partial \bar z} d\bar z \wedge dz$$

Then maps in diagram are:

$$\begin{tikzcd} \Omega^{0,1} \arrow[r, "\partial"] & \Omega^2 \\ \Omega^0 \arrow[u, "\bar\partial"] \arrow[r, "\partial"'] & \Omega^{1,0} \arrow[u, "\bar\partial"'] \end{tikzcd}$$

Finally if $S \subset X$ is compact surface-with-boundary

α = holomorphic 1-form on nbhd. of S. Then α is closed,

so by Stokes' thm, $\displaystyle\int_{\partial S} \alpha = 0$.

One version of Cauchy's theorem on Riemann surface.

Also define meromorphic 1-form in obvious way. Use Cauchy's thm to prove that sum of residues over all poles of compact R.S. is 0.