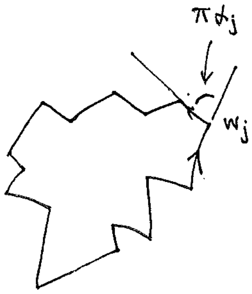
Schwarz-Christoffel formula

(Do the case of upper half plane to polygon.)

polygon: simply-connected, bounded region Ω of $\mathbb{C}$ with boundary consisting of n straight line segments has closure called "n-gon"



Exterior angles: extend line segments to rays according to positive orientation

Theorem: Let $w_1, \dots, w_n$ be vertices of n-gon. with "exterior angles" $\pi\alpha_i$, $i = 1, \dots, n$ where $\alpha_i \in (-1, 1)$

then conformal maps $f : \mathcal{H}$: upper half plane $= \{ z \mid \operatorname{Im}(z) > 0 \}$

$\to \Omega$: interior of n-gon

can be represented in ~~have~~ the form:

$$f(z) = c_0 \int_{z_0}^{z} (\xi - x_1)^{-\alpha_1} \cdots (\xi - x_{n-1})^{-\alpha_{n-1}} \, d\xi + c_1$$

where $c_0 \neq 0$, c_1 are constants.

$z_0 \in \mathcal{H}$, integration taken along any path from z_0 to z

x_i on real line mapping to w_i : vertices of polygon.

(and $\{\infty\}$ maps to w_n)

Remark: One can show, using theorems about boundaries in Ahlfors, that f extends to a continuous map from $\overline{\mathcal{H}} = \{ z \mid \operatorname{Im}(z) \geq 0 \}$ to the n-gon. (can't be analytic since it doesn't preserve angles)

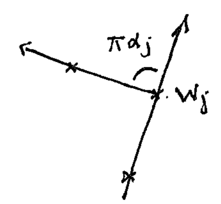
Let $g(\xi)$ be the integrand appearing in $f(z)$, so $f'(z) = g(z)$

<u>Idea</u>: $\arg f'(z)$ <u>constant</u> along segments of the polygon

And indeed $\arg g(z) = \arg(c_0) - \alpha_1 \arg(z - x_1) - \dots - \alpha_{n-1} \arg(z - x_{n-1})$

So as we traverse the real axis in z, and z passes x_j then the $\arg(z - x_j)$ changes from π to 0, i.e. $\arg(g(z))$ changes by $\alpha_j \cdot \pi$.

Indeed this was precisely the exterior angle of the n-gon:



As we pass the point x_{n-1}, then the $\arg g(z)$ has increased to $\arg(c_0)$ and finally angle $\pi\alpha_n$ is determined by condition that angles on exterior of polygon sum to 2π.

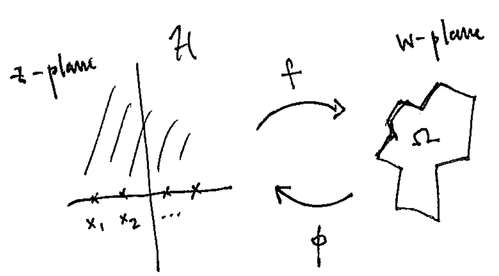
position of x_j's determines the length of the segments of the polygon.

(special case of triangle, then angles determine it, so x_1, x_2 may be chosen arbitrarily. Same is true in general: pick two points arbitrarily.)

c_0, c_1 scale, rotate, translate the polygon.

<u>Difficult part</u>: showing all such maps ~~have~~ are expressible in form of such integrals.

very rough <u>outline</u>: use reflection principle (do reflection of domain in line, arc of circle and extend holomorphic function. Means of analytic continuation)



to construct meromorphic function of $\mathbb{C}$ with poles at $x_1, \dots, x_{n-1}$, $g \to 0$ as $z \to \infty$

entire function made by analyzing Laurent exp.

use Liouville's thm.

Takes <u>form</u>: $h(z) + \sum_{j=1}^{n-1} \frac{\alpha_j}{z - x_j}$

where $\frac{d}{dz}\left(\log(f'(z))\right) = h(z)$ solve for f.

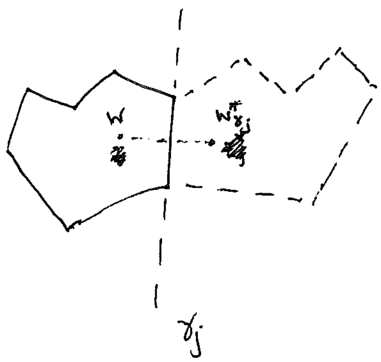
Go just a bit further. - we just need reflection in a line.

Line segment γ_j connecting w_{j-1}-vertex to w_j-vertex, then

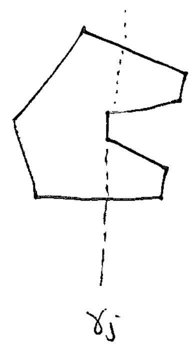
reflection in γ_j: $\quad w \longmapsto w^*_{\gamma_j} := \dfrac{w_j - w_{j-1}}{\overline{w_j} - \overline{w_{j-1}}}\left(\bar{w} - \overline{w_{j-1}}\right) + w_{j-1}$

Better to change z's to w's here to reflect earlier notation: Polygon in w plane

ok



nice



not nice (issue: convexity of polygon)
geometrically, but
definition of analytic extension ok. (check this!)

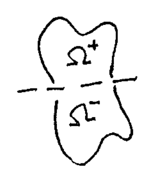
Give conditions under which analytic function ϕ on Ω can be extended to

$\Omega \cup \gamma_j \cup \underset{\sim}{\Omega^*}$
image under reflection.

<u>Motivation</u>: $f(z)$ analytic in Ω, then $\overline{f(\bar{z})}$ analytic on Ω^*: reflection across real axis $z \mapsto \bar{z}$ of Ω.

Suppose Ω symmetric w.r.t. real axis.

e.g. Ω



and f real-valued on some open interval of real axis.

Then for these real values $f(z) = \overline{f(\bar{z})}$, and hence for all $z \in \Omega$ since analytic functions det'd by values in an open set.

Symmetry principle is stronger: if only know

f analytic on Ω^+, continuous on real line $\cap\, \Omega$, and real-valued in interval then f has analytic continuation by setting $f(z) = \overline{f(\bar{z})}$ for $z \in \Omega^-$.

use mean value prop. to analyze on real line. See Ahlfors 4.6

the issue in our example is that we have analytic continuation along each side of the polygon, so a uniform definition on $\mathcal{H}$, but many different extensions to $\mathcal{H}^-$.

Compose two reflections $w^*_{\gamma_j} \longmapsto w \longmapsto w^*_{\gamma_k} = c_{jik}\, w^*_{\gamma_j} + d_{jik}$

$c_{jik} \neq 0$, d_{jik} constants

so that on $z \in \mathcal{H}^-$,

$$\tilde{f}_k = c_{jik}\tilde{f}_j + d_{jik}$$

$\tilde{f}_j, \tilde{f}_k$ are respective extensions of f.

$$\Rightarrow \quad \frac{\tilde{f}_k''}{\tilde{f}_k'} = \frac{\tilde{f}_j''}{\tilde{f}_j'} \quad \text{for all } z \in \mathcal{H}^-.$$

i.e. $\exists$ single valued extension of $\dfrac{f''}{f'}$ on $\mathbb{C} \cup \{\infty\} \setminus \{x_1, \dots, x_{n-1}\}$.

Now study residues at x_i...