Math 8702 – Spring 2014 – Problem Set 1

1. (Ahlfors, 2.1.2, p. 28, #3) Find the most general harmonic polynomial of the form

$$ax^3 + bx^2y + cxy^2 + dy^3.$$

Determine the harmonic conjugate. (See p. 27 of Ahlfors for an alternate method for determining the conjugate harmonic function in this case.)

2. (Ahlfors, 2.1.2, p. 28, #6) Prove that a function $u(z)$ is harmonic if and only if $u(\overline{z})$ is harmonic.

3. Determine the subset of the plane on which the following functions are harmonic:

a) $u_1(x, y) = \mathrm{Im}(z + 1/z)$

b) $u_2(x, y) = \frac{y}{(x-1)^2+y^2}$

4. Let u be the harmonic function on the upper half plane ($y > 0$) defined by

$$u(x, y) = 1 - y + \frac{x}{x^2 + y^2}.$$

Find a corresponding harmonic function in the first quadrant under the map $z \mapsto z^2$.

5. Does there exist a bijective, conformal map between a simply connected region and one that is not simply connected?