Name: ____________________

Problem Set 6
Math 4281, Fall 2013
Due: Friday, October 18

Read Sections 21.1 (stop after Example 1), 21.2 (stop after Example 11), and 16.3, as well as Theorem 17.3 in your textbook.

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1. Prove that $\mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\sqrt{2} + i)$, but $\mathbb{Q}(\sqrt{2}i) \subsetneq \mathbb{Q}(\sqrt{2}, i)$.
2. Find the splitting field for the following polynomials in $\mathbb{Q}[x]$:
   a. $f(x) = x^6 - 1$
   b. $f(x) = x^4 - 10x^2 + 1$ (Hint: $\sqrt{2} + \sqrt{3}$ is one root.)
3. Suppose that $\alpha \in \mathbb{C}$ is a root of $f(x) \in \mathbb{Q}[x]$. Find the multiplicative inverse of $\beta \in \mathbb{Q}(\alpha)$. (Hint: Use the Euclidean algorithm.)
   a. $f(x) = x^2 + 3x - 3 \in \mathbb{Q}[x],\ \beta = \alpha - 1$
   b. $f(x) = x^3 + x^2 + 2x + 1 \in \mathbb{Q}[x],\ \beta = \alpha^2 + 1$
4. Let R be a commutative ring with 1.
   a. Prove that if $I \subseteq R$ is an ideal and $1 \in I$, then $I = R$.
   b. Prove that $a \in R$ is a unit if and only if $\langle a \rangle = R$.
   c. Prove that the only ideals in R are $\langle 0 \rangle$ and R if and only if R is a field.
5. Find all ideals in $\mathbb{Z}$ and in $\mathbb{Z}_6$.
6. Prove that $\phi\colon \mathbb{Z}_p \to \mathbb{Z}_p$, $\phi(a) = a^p$, is a ring homomorphism.
7. Give the addition and multiplication tables of $\mathbb{Z}_2[x]/\langle x^3 + x + 1 \rangle$.
8. Find all ring homomorphisms:
   a. $\phi\colon \mathbb{Z}_2 \to \mathbb{Z}$
   b. $\phi\colon \mathbb{Z}_2 \to \mathbb{Z}_6$
   c. $\phi\colon \mathbb{Z}_6 \to \mathbb{Z}_2$

Through the course of this assignment, I have followed the guidelines of the University of Minnesota Student Conduct Code.

Signed: ____________________