Name: ______________________________

Problem Set 7

Math 4281, Fall 2013

Due: Friday, October 25

Read Sections 16.3, 21.1 (thru Thm. 21.2), and 21.2 (thru Thm. 21.17), in your textbook.

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1. Let R be a commutative ring with 1, and let $I, J \subset R$ be ideals. Define
$$I \cap J = \{a \in R \mid a \in I \text{ and } a \in J\} \quad \text{and} \quad I + J = \{a + b \in R \mid a \in I, b \in J\}.$$
   a. Prove that $I \cap J$ and $I + J$ are ideals.
   b. Suppose $R = \mathbb{Z}$ or $F[x]$ for a field F, $I = \langle a \rangle$, and $J = \langle b \rangle$. Identify $I \cap J$ and $I + J$ in terms of a and b.
   c. Let $a_1, \ldots, a_n \in R$. Prove that $\langle a_1, \ldots, a_n \rangle = \langle a_1 \rangle + \cdots + \langle a_n \rangle$.

2. a. Prove that the function $\phi\colon \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ defined by $\phi(a + b\sqrt{2}) = a - b\sqrt{2}$ is a ring isomorphism.
   b. Define the function $\phi\colon \mathbb{Q}(\sqrt{3}) \to \mathbb{Q}(\sqrt{7})$ by $\phi(a + b\sqrt{3}) = a + b\sqrt{7}$. Is ϕ a ring isomorphism? Is there any isomorphism between these rings?

3. Establish the following isomorphisms by using the Fundamental Homomorphism Theorem:
   a. $\mathbb{R}[x]/\langle x^2 + 6 \rangle \cong \mathbb{C}$
   b. $\mathbb{Q}[x]/\langle x^2 + x + 1 \rangle \cong \mathbb{Q}(\sqrt{3}i)$
   c. $\mathbb{Z}_3 \times \mathbb{Z}_4 \cong \mathbb{Z}_{12}$

4. a. Prove that the composition of two ring isomorphisms is a ring isomorphism.
   b. Suppose that $\phi\colon R \to S$ is a ring isomorphism. Prove that the inverse function $\phi^{-1}\colon S \to R$ is a ring homomorphism (and therefore also an isomorphism).

5. Let F be a field, $f(x) \in F[x]$, and K be a field extension of F containing the root α of $f(x)$.
   a. If $\sigma\colon K \to K$ is a ring isomorphism with the property that $\sigma(a) = a$ for all $a \in F$, show that $\sigma(\alpha)$ is likewise a root of $f(x)$.
   b. Apply (a) to show that the complex roots of a real polynomial occur in conjugate pairs.
   c. Apply (a) to show that if $n \in \mathbb{N}$ is not a perfect square, and $\sqrt{n}$ is a root of $f(x) \in \mathbb{Q}[x]$, then $-\sqrt{n}$ is a root as well.

6. Answer, giving proofs or disproofs.
   a. Is $\mathbb{Z}_2[x]/\langle x^2 \rangle \cong \mathbb{Z}_4$, or is $\mathbb{Z}_2[x]/\langle x^2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$?
   b. Is $\mathbb{Z}_3[x]/\langle x^2 - 1 \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$?
   c. Is $\mathbb{Q}[x]/\langle x^2 - 1 \rangle \cong \mathbb{Q} \times \mathbb{Q}$?

Through the course of this assignment, I have followed the guidelines of the University of Minnesota Student Conduct Code.

Signed: ______________________________