Name: ______________________

# Problem Set 7
## Math 4281, Spring 2014
Due: Wednesday, March 12

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**Ring homomorphisms and ideals**

1. Find all ring homomorphisms:

   a. $\phi\colon \mathbb{Z}_2 \to \mathbb{Z}$

   b. $\phi\colon \mathbb{Z}_2 \to \mathbb{Z}_6$

   c. $\phi\colon \mathbb{Z}_6 \to \mathbb{Z}_2$

2. Prove that if $p$ is prime and $\phi\colon \mathbb{Z}_p \to \mathbb{Z}_p$, $\phi(a) = a^p$, is a ring homomorphism.

3. Find all ideals in $\mathbb{Z}$ and in $\mathbb{Z}_6$.

4. Let $R$ be a commutative ring with $1$, and let $a_1, \ldots, a_n \in R$. Show that

$$\langle a_1, \ldots, a_n \rangle := \{r_1 a_1 + \cdots + r_n a_n \mid r_i \in R \text{ for all } i\} \subseteq R$$

   is an ideal in $R$.

5. Let $R$ be a commutative ring with $1$, and let $I, J \subset R$ be ideals. Define

$$I \cap J = \{a \in R \mid a \in I \text{ and } a \in J\} \quad \text{and} \quad I + J = \{a + b \in R \mid a \in I, b \in J\}.$$

   a. Prove that $I \cap J$ and $I + J$ are ideals.

   b. Suppose $R = \mathbb{Z}$ or $F[x]$ for a field $F$, $I = \langle a \rangle$, and $J = \langle b \rangle$. Identify $I \cap J$ and $I + J$ in terms of $a$ and $b$.

   c. Let $a_1, \ldots, a_n \in R$. Prove that $\langle a_1, \ldots, a_n \rangle = \langle a_1 \rangle + \cdots + \langle a_n \rangle$.

6. Let $R$ be a commutative ring with $1$.

   a. Prove that if $I \subseteq R$ is an ideal and $1 \in I$, then $I = R$.

   b. Prove that $a \in R$ is a unit if and only if $\langle a \rangle = R$.

   c. Prove that the only ideals in $R$ are $\langle 0 \rangle$ and $R$ if and only if $R$ is a field.



Signed: ______________________