Name: ______________________

Problem Set 9
Math 4281, Spring 2014
Due: Wednesday, April 2

Ring isomorphisms

1. Establish the following isomorphisms by using the Fundamental Homomorphism Theorem:
   a. $\mathbb{R}[x]/\langle x^2+6\rangle \cong \mathbb{C}$
   b. $\mathbb{Q}[x]/\langle x^2+x+1\rangle \cong \mathbb{Q}(\sqrt{3}i)$
   c. $\mathbb{Z}_3 \times \mathbb{Z}_4 \cong \mathbb{Z}_{12}$

2. Let F be a field, $f(x) \in F[x]$, and K be a field extension of F containing the root α of $f(x)$.
   a. If $\sigma\colon K \to K$ is a ring isomorphism with the property that $\sigma(a) = a$ for all $a \in F$, show that $\sigma(\alpha)$ is likewise a root of $f(x)$.
   b. Apply (a) to show that the complex roots of a real polynomial occur in conjugate pairs.
   c. Apply (a) to show that if $n \in \mathbb{N}$ is not a perfect square, and $\sqrt{n}$ is a root of $f(x) \in \mathbb{Q}[x]$, then $-\sqrt{n}$ is a root as well.

Vector spaces and field extensions

3. Prove that the real numbers 1 and $\sqrt{3}$ are linearly independent over $\mathbb{Q}$. Do the same for 1, $\sqrt{3}$, and $\sqrt{5}$.

4. Give a basis for each of the given vector spaces over the given field. What is the degree of each field extension?
   a. $\mathbb{Q}(\sqrt{3}, i)$ over $\mathbb{Q}$
   b. $\mathbb{Q}(\sqrt{3}, i)$ over $\mathbb{Q}(i\sqrt{3})$
   c. $\mathbb{Z}_2[x]/\langle x^3+x+1\rangle$ over $\mathbb{Z}_2$
   d. $\mathbb{Q}(\sqrt[5]{8})$ over $\mathbb{Q}$

5. Let F be a field. Suppose that K is a field extension of F of finite degree. Prove that if $\alpha \in K$, then there is an irreducible polynomial $f(x) \in F[x]$ having α as a root. (Hint: If $[K : F] = n$, consider $1, \alpha, \alpha^2, \dots, \alpha^n$.)

Throughout the course of this assignment, I have followed the guidelines of the University of Minnesota Student Conduct Code.

Signed: ______________________