Math 5385 - Spring 2018
Problem Set 11

Submit solutions to **three** of the following problems.

1. Consider the ideal $I := \langle x^2, xy \rangle$ in $\mathbb{C}[x,y]$. For any $c \in \mathbb{C}$, prove that $I = \langle x \rangle \cap \langle x^2, y - cx \rangle$ is an irredundant primary decomposition of I.

2. Let I be a monomial ideal in $S := \Bbbk[x_1, \dots, x_n]$.
   (a) Suppose that x^u is a minimal generator of I such that $x^u = x^{v_1} x^{v_2}$, where the monomials x^{v_1} and x^{v_2} are relatively prime. Show that
   $$I = (I + \langle x^{v_1} \rangle) \cap (I + \langle x^{v_2} \rangle).$$
   (b) Find an irredundant primary decomposition of $\langle x^3y, x^3z, xy^3, y^3z, xz^3, yz^3 \rangle$.

3. A homogeneous polynomial $f \in \Bbbk[x_0, \dots, x_n]$ can also be used to define the *affine* variety $C = V_a(f) \subset \mathbb{A}^{n+1}(\Bbbk) = \Bbbk^{n+1}$. We call C the *affine cone* over the projective variety $X = V(f) \subset \mathbb{P}^n$.
   (a) Show that if C contains the point $(a_0, \dots, a_n) \neq (0, \dots, 0)$, then C contains the whole line through the origin in $\mathbb{A}^{n+1}(\Bbbk)$ spanned by $(a_0, \dots, a_n)$.
   (b) Consider the point $[a_0 : \cdots : a_n] \in \mathbb{P}^n$ with homogeneous coordinates $(a_0, \dots, a_n)$. Show that $[a_0 : \cdots : a_n]$ is in the projective variety X if and only if the line through the origin determined by $(a_0, \dots, a_n)$ is contained in C.
   (c) Deduce that C is the union of the collection of lines through the origin in $\mathbb{A}^{n+1}(\Bbbk)$ corresponding to the points in X.

4. In this problem, we study how lines in $\mathbb{R}^n$ relate to points in $\mathbb{P}^n(\mathbb{R}) = \mathbb{R}^n \cup \mathbb{P}^{n-1}(\mathbb{R})$. Given a line L in $\mathbb{R}^n$, we can parametrize L by the formula $a + bt$, where $a \in L$ and b is a nonzero vector parallel to L. In coordinates, we write this parametrization as $(a_1 + b_1 t, \cdots, a_n + b_n t)$.
   (a) Regard L as lying in $\mathbb{P}^n(\mathbb{R})$ via the homogeneous coordinates
   $$[1 : a_1 + b_1 t : \cdots : a_n + b_n t].$$
   To find out what happens as $t \to \pm\infty$, divide by t to obtain
   $$\left[\frac{1}{t} : \frac{a_1}{t} + b_1 : \cdots : \frac{a_n}{t} + b_n\right].$$
   What are the coordinates for the point $L \cap \mathbb{P}^{n-1}(\mathbb{R})$ in $H = \mathbb{P}^{n-1}(\mathbb{R})$?
   (b) The line L has many parametrizations. Show $L \cap \mathbb{P}^{n-1}(\mathbb{R})$ is the same for all parametrizations of L. **Hint.** Two nonzero vectors are parallel if and only if they are scalar multiples of each other.
   (c) Parts (a) and (b) show that a line L in $\mathbb{R}^n$ has a well-defined point at infinity in $H = \mathbb{P}^{n-1}(\mathbb{R})$. Show that two lines in $\mathbb{R}^n$ are parallel if and only if they have the same point at infinity in $\mathbb{P}^n(\mathbb{R})$.