Math 5385 - Spring 2018
Problem Set 12

Submit solutions to **three** of the following problems.

1. This problem studies some properties of the ideal $I_0 = \langle x_0, \ldots, x_n \rangle \subseteq \mathbb{k}[x_0, \ldots, x_n]$.
   (a) Show that every proper homogeneous ideal in $\mathbb{k}[x_0, \ldots, x_n]$ is contained in I_0.
   (b) Show that the r-th power I_0^r is the ideal generated by the collection of monomials in $\mathbb{k}[x_0, \ldots, x_n]$ of total degree exactly r and deduce that every homogeneous polynomial of degree $\geq r$ is in I_0^r.
   (c) Let $V = V(I_0) \subseteq \mathbb{P}^n(\mathbb{k})$ and $C_V = V_a(I_0) \subseteq \mathbb{A}^{n+1}(\mathbb{k}) = \mathbb{k}^{n+1}$. Show that $I_a(C_V) \neq I(V)$, and explain why this does not contradict equation (2) in the proof of Theorem 8.3.9 in the text.

2. A homogeneous ideal is *prime* if it is prime as an ideal in $\mathbb{k}[x_0, \ldots, x_n]$.
   (a) Show that a homogeneous ideal $I \subseteq \mathbb{k}[x_0, \ldots, x_n]$ is prime if and only if whenever the product of two *homogeneous* polynomials F, G satisfies $FG \in I$, then $F \in I$ or $G \in I$.
   (b) Let $\mathbb{k}$ be algebraically closed. Let I be a homogeneous ideal. Show that the projective variety $V(I)$ is irreducible if I is prime. When I is radical, prove that the converse holds, i.e., that I is prime if $V(I)$ is irreducible. **Hint.** Consider the proof of the corresponding statement in the affine case (Proposition 4.5.3).

3. The twisted cubic is one member of an infinite family of curves known as the *rational normal curves.* The rational normal curve in $\mathbb{A}^n(\mathbb{k}) = \mathbb{k}^n$ is the image of the polynomial parametrization $\varphi\colon \mathbb{k} \to \mathbb{A}^n(\mathbb{k})$ given by $\varphi(t) = (t, t^2, t^3, \ldots, t^n)$.

   By our results on implicitization from Chapter 3, the rational normal curves are affine varieties. Their projective closures in $\mathbb{P}^n$ are also known as rational normal curves.
   (a) Find affine equations for the rational normal curves in $\mathbb{A}^4(\mathbb{k})$ and $\mathbb{A}^5(\mathbb{k})$.
   (b) Homogenize your equations from part (a) and consider the projective varieties defined by these homogeneous polynomials. Do your equations define the projective closure of the affine curve? Are there any "extra" components at infinity?
   (c) Using Theorems 8.4.4 and 8.4.8, find a set of homogeneous equations defining the projective closures of these rational normal curves in $\mathbb{P}^4$ and $\mathbb{P}^5$, respectively.
   (d) Show that the rational normal curve in $\mathbb{P}^n$ is the variety defined by the set of homogeneous quadrics obtained by taking all possible 2×2 subdeterminants of the $2 \times n$ matrix:
   $$\begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} \\ x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix}$$

4. The Cartesian product of two affine spaces is simply another affine space: $k^m \times k^n = k^{m+n}$. If we use the standard inclusions $k^n \subseteq \mathbb{P}^n$, $k^m \subseteq \mathbb{P}^m$, and $k^{n+m} \subseteq \mathbb{P}^{n+m}$ given by Proposition 8.2.2 (i.e., zeroth coordinate equal to 1), how is $\mathbb{P}^{n+m}$ different from $\mathbb{P}^n \times \mathbb{P}^m$ (as a set)?