Math 5385 - Spring 2018
Problem Set 4

Submit solutions to **three** of the following problems.

1. A ring R satisfies the *descending chain condition* if any descending sequence of ideals in R stabilizes, i.e. $I_1 \supset I_2 \supset I_3 \supset \cdots$ implies $I_j = I_{j+1}$ for $j \gg 0$. Such a ring is *Artinian.*
   (a) Show that $\mathbb{Z}/n\mathbb{Z}$ and $k[x]/\langle x^n \rangle$ are Artinian.
   (b) Show that $\mathbb{Z}$ and $k[x]$ are not Artinian.
   (c) Show that every prime ideal in an Artinian ring is maximal.
   Hint. If $J \supset I$, then consider $J \supset J^2 \supset J^3 \supset \cdots \supset I$.

2. Let $I := \langle wy - x^2, wz - xy, xz - y^2 \rangle \subset \mathbb{Q}[x, y, z, w]$.
   (a) Find the reduced Gröbner basis of I with respect to the graded reverse lexicographic order and $x > y > z > w$.
   (b) Find the reduced Gröbner basis of I with respect to the lexicographic order and $x > y > z > w$.
   (c) *(Bonus)* The ideal I has eight distinct leading term ideals; can you list these eight monomial ideals?

3. Fix the lexicographic order order on $S = \Bbbk[x_1, \ldots, x_n]$ with $x_1 > \cdots > x_n$. Let $A = [a_{i,j}]$ be an $m \times n$ matrix with entries in $\Bbbk$ and let $f_i = a_{i,1}x_1 + \cdots + a_{i,n}x_n$ be the linear polynomials in S determined by the rows of A. Suppose that $B = [b_{i,j}]$ is the row-reduced echelon matrix determined by A and let $g_1, \ldots, g_r$ be the linear polynomials determined by the nonzero rows in B.
   (a) Prove that $\langle f_1, \ldots, f_m \rangle = \langle g_1, \ldots, g_r \rangle$.
   (b) Show that $g_1, \ldots, g_r$ form a Gröbner basis of $\langle f_1, \ldots, f_m \rangle$. (See hint on p.96.)
   (c) Explain why $g_1, \ldots, g_r$ is the reduced Gröbner basis.

4. Suppose we have n points $V = \{(a_1, b_1), \ldots, (a_n, b_n)\} \subseteq \Bbbk^2$, where $a_1, \ldots, a_n$ are distinct. The *Lagrange interpolation polynomial* for these points is

$$h(x) := \sum_{i=1}^{n} b_i \sum_{j \neq i} \frac{x_j - a_j}{a_i - a_j} \in \Bbbk[x].$$

   (a) Show that $h(a_i) = b_i$ for $i = 1, \ldots, n$ and explain why h has degree $\leq n - 1$.
   (b) Prove that $h(x)$ is the unique polynomial of degree $\leq n-1$ satisfying $h(a_i) = b_i$ for $i = 1, \ldots, n$.
   (c) Prove that $I(V) = \langle f(x), y - h(x) \rangle \subseteq \Bbbk[x, y]$, where $f(x) := \prod_{i=1}^{n} (x - a_i)$.
   Hint. Divide $g \in I(V)$ by $f(x), y - h(x)$ using lex order with $y > x$.
   (d) Prove that $\{f(x), y - h(x)\}$ is the reduced Gröbner basis for $I(V) \subseteq \Bbbk[x, y]$ for lex order with $y > x$.