Math 5385 - Spring 2018
Problem Set 5

Submit solutions to **three** of the following problems.

1. Consider the ideal $I = \langle x^2 + y^2 + z^2 + 2, 3x^2 + 4y^2 + 4z^2 + 5\rangle$. Let $X = V(I)$, let $\pi\colon \mathbb{A}^3(\Bbbk) \to \mathbb{A}^2(\Bbbk)$ be the projection given by $(x, y, z) \mapsto (y, z)$, and let $J = I \cap \Bbbk[y, z]$.
   (a) If $\Bbbk = \mathbb{C}$, then prove that $V(J) = \pi(X)$.
   (b) If $\Bbbk = \mathbb{R}$, then prove that $X = \varnothing$ and $V(J)$ is infinite. Hence, $V(J)$ may be much larger than the smallest affine variety containing $\pi(X)$ when the field is not algebraically closed.

2. Use elimination to solve the system:

$$0 = x^2 + 2y^2 - y - 2z, \qquad 0 = x^2 - 8y^2 + 10z - 1, \qquad 0 = x^2 - 7yz.$$

   How many solutions are there in $\mathbb{A}^3(\mathbb{R})$; how many are there in $\mathbb{A}^3(\mathbb{C})$?

3. Let $\Bbbk$ be an infinite field.
   (a) Consider

$$A := \begin{bmatrix} 1&1&1&1&0&0&0&0 \\ 0&0&0&0&1&1&1&1 \\ 1&1&0&0&1&1&0&0 \\ 0&0&1&1&0&0&1&1 \\ 1&0&1&0&1&0&1&0 \\ 0&1&0&1&0&1&0&1 \end{bmatrix}.$$

   Find a generating set for the toric ideal I_A, which is the ideal for the smallest variety containing the image of the map $\varphi\colon \mathbb{A}^6(\Bbbk) \to \mathbb{A}^8(\Bbbk)$ given by

$$\varphi(t_1, \dots, t_6) \mapsto (t_1t_3t_5, t_1t_3t_6, t_1t_4t_5, t_1t_4t_6, t_2t_3t_5, t_2t_3t_6, t_2t_4t_5, t_2t_4t_6).$$

   (b) Find the equations for the image of the rational map $\rho\colon \mathbb{A}^4(\Bbbk) \to \mathbb{A}^6(\Bbbk)$ defined by

$$\rho(x_1, x_2, x_3, x_4) = \left(\frac{1}{x_1x_2}, \frac{1}{x_1x_3}, \frac{1}{x_1x_4}, \frac{1}{x_2x_3}, \frac{1}{x_2x_4}, \frac{1}{x_3x_4}\right).$$

4. Let $g \in \Bbbk[t]$ be a polynomial such that $g(0) = 0$. Assume $\mathbb{Q} \subseteq \Bbbk$.
   (a) Prove that $t = 0$ is a root of multiplicity ≥ 2 of g if and only if $g'(0) = 0$.
   Hint. Write $g(t) = th(t)$ and use the product rule.
   (b) More generally, prove that $t = 0$ is a root of multiplicity $\geq m$ if and only if $g'(0) = g''(0) = \cdots = g^{(m-1)}(0) = 0$.