Math 5385 - Spring 2018
Problem Set 9

Submit solutions to **three** of the following problems.

1. Let I and J be ideals in $S = \Bbbk[x_1, \ldots, x_n]$, where $\Bbbk$ is an arbitrary field.
   (a) If $I^m \subseteq J$ for some integer $m > 0$, then show that $\sqrt{I} \subseteq \sqrt{J}$.
   (b) Prove that $\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$.
2. Two ideals I and J in $S := \Bbbk[x_1, \ldots, x_n]$ are comaximal if and only if $I + J = S$.
   (a) If $\Bbbk$ is an algebraically closed field, then show that I and J are comaximal if and only if $V(I) \cap V(J) = \varnothing$. Give an example to show that this is false in general.
   (b) If I and J are comaximal, then show that $IJ = I \cap J$.
   (c) If I and J are comaximal, then show that I^i and J^j are comaximal for all positive integers i and j.
3. Let I, J be ideals in $\Bbbk[x_1, \ldots, x_n]$ and suppose that $I \subseteq \sqrt{J}$. Show that $I^m \subseteq J$ for some integer $m \geq 1$. **Hint.** You will need to use the Hilbert Basis Theorem.
4. Find the Zariski closure of the following sets:
   (a) The projection of the hyperbola $V(xy - 1)$ in $\mathbb{R}^2$ onto the x-axis.
   (b) The boundary of the first quadrant in $\mathbb{R}^2$.
   (c) The set $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$.
5. Let $I, J, K \subseteq \Bbbk[x_1, \ldots, x_n]$ be ideals. Prove the following:
   (a) $IJ \subseteq K$ if and only if $I \subseteq K : J$.
   (b) $(I : J) : K = I : JK$.