BIG I-FUNCTIONS

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To Professor Shigeru Mukai, on the occasion of his 60^{th} *birthday.*

Abstract. We introduce a new big I-function for certain GIT quotients $W/\!\!/\mathbf{G}$ using the quasimap graph space from infinitesimally pointed $\mathbb{P}^1$ to the stack quotient $[W/\mathbf{G}]$. This big I-function is expressible by the small I-function introduced in [6, 10]. The I-function conjecturally generates the Lagrangian cone of Gromov-Witten theory for $W/\!\!/\mathbf{G}$ defined by Givental. We prove the conjecture when $W/\!\!/\mathbf{G}$ has a torus action with good properties.

1. Introduction

Let X be a nonsingular quasi-projective variety with a torus $\mathbf{T}$-action such that the $\mathbf{T}$-fixed locus $X^{\mathbf{T}}$ is projective. We allow $\mathbf{T}$ the trivial group. The $\mathbf{T}$-equivariant rational Gromov-Witten theory for X is encoded in the genus 0 prepotential F, i.e., the generating function of gravitational Gromov-Witten invariants defined by the integration of psi-classes and pullbacks of cohomology classes of target X against the virtual fundamental classes of the moduli space of k pointed, genus 0, numerical class β stable maps to X.

Givental shows that the graph of the formal 1-form dF is a Lagrangian cone in a suitably defined infinite dimensional symplectic space and the cone is generated by the J-function (see [16]). The big J-function for X is a generating function of genus 0 GW-invariants with gravitational insertions at one point, and any number of primary insertions. It is a difficult problem to compute the J-function in general. In the case when X has a GIT presentation $X = W/\!\!/\mathbf{G}$ with W affine, there is a replacement of the J-function. It is the so-called I-function, introduced in [6, 10] as a generalization of Givental's small I-function for toric targets. While it is shown in [7] that I and J are related via generalized Mirror Theorems, the big I-function is equally difficult to compute. The purpose of this paper is to remedy this situation by introducing a new version of I-functions (for the same kind of GIT targets). This new function, which we denote by $\mathbb{I}$, can be computed explicitly in closed form in many cases, and the J-function is obtained from it via the Birkhoff factorization procedure, as given in [13].

The precise GIT set-up is as follows. Let W be an affine variety with a linear right action of a reductive algebraic group $\mathbf{G}$. For any rational character θ of $\mathbf{G}$, denote by $W^{ss}(\theta)$ the semistable locus of W with respect to θ. Assume that $W^{ss}(\theta)$ is nonsingular, W has at worst l.c.i singularities, and $\mathbf{G}$ acts on $W^{ss}(\theta)$ freely (however, see [5] for allowing finite non-trivial stabilizers).

Given such a triple $(W, \mathbf{G}, \theta)$, there is a relative compactification of the space of maps from $\mathbb{P}^1$ to $W/\!\!/\mathbf{G}$ of given numerical class β (see Definition 2.1 for the notion of numerical class), keeping the domain curve $\mathbb{P}^1$ but allowing maps $\mathbb{P}^1 \to [W/\mathbf{G}]$ to the stack quotient. The "compactification" is called the quasimap graph space and defined to be

$$QG_{0,0,\beta}(W/\!\!/\mathbf{G}) := \{f \in \mathrm{Hom}(\mathbb{P}^1, [W/\mathbf{G}]) : f^{-1}(W/\!\!/\mathbf{G}) \neq \emptyset, \beta_f = \beta\}.$$

It is an algebraic space proper over the affine quotient $W/_{\mathrm{aff}}\mathbf{G}$ (see [10]). This graph space is equipped with a $\mathbb{C}^*$-action induced from the $\mathbb{C}^*$-action on $\mathbb{P}^1$, as well as with a natural equivariant perfect obstruction theory. There is a distinguished open and closed subspace F_β of the $\mathbb{C}^*$-fixed locus of the graph space $QG_{0,0,\beta}(W/\!\!/\mathbf{G})$. The small I-function is defined by the localization residue at F_β as follows:

$$\mathbb{I}(q,z) := \sum_\beta q^\beta (ev_\bullet)_*(\mathrm{Res}_{F_\beta}[QG_{0,0,\beta}(W/\!\!/\mathbf{G})]^{\mathrm{vir}}),$$

where $ev_\bullet$ is the evaluation map from F_β to $W/\!\!/\mathbf{G}$ at the generic point of $\mathbb{P}^1$ and z is the $\mathbb{C}^*$-equivariant parameter. The sum is over all θ-effective "curve classes" $\beta \in \mathrm{Eff}(W, \mathbf{G}, \theta)$, see Definition 2.8 for the notion of θ-effective class..

There is another evaluation map $\hat{ev}_\beta$ from F_β at $0 \in \mathbb{P}^1$. The codomain of $\hat{ev}_\beta$ is the stack quotient $[W/\mathbf{G}]$. Therefore we have

$$[W/\mathbf{G}] \xleftarrow{\hat{ev}_\beta} F_\beta \xrightarrow{ev_\bullet} W/\!\!/\mathbf{G}\ .$$

If we write $\mathbb{I}(q,z) = \sum_\beta q^\beta I_\beta(q,z)$, then the big I-function in this paper is

$$\mathbb{I}(\mathbf{t}) = \sum_\beta q^\beta (ev_\bullet)_*(\exp(\hat{ev}_\beta^*(\mathbf{t})/z) \cap \mathrm{Res}_{F_\beta}[QG_{0,0,\beta}(W/\!\!/\mathbf{G})]^{\mathrm{vir}}),$$

for $\mathbf{t} \in H^*([W/\mathbf{G}], \mathbb{Q})$.

We conjecture that $\mathbb{I}(\mathbf{t})$ is on the Lagrangian cone of Gromov-Witten theory of $W/\!\!/\mathbf{G}$ with Novikov variables from $\mathrm{Eff}(W, \mathbf{G}, \theta)$. We prove the conjecture when there is an action by a torus $\mathbf{T}$ on W, commuting with $\mathbf{G}$-action, and such that $X = W/\!\!/\mathbf{G}$ has only isolated 0 and 1-dimensional $\mathbf{T}$-orbits.

To prove the conjecture, we introduce the stable quasimaps with $\epsilon := (1, ..., 1, \varepsilon, ..., \varepsilon)$-weighted markings and the J^ϵ-function whose special case is the I-function. The proof is parallel to the proof of the corresponding theorems in [7].

In the last section we explain how to obtain an explicit closed formula for the big $\mathbb{I}(\mathbf{t})$ for toric varieties and for complete intersections in them.

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2. Weighted Stable Quasimaps

Throughout the paper the base field is $\mathbb{C}$.

2.1. **θ-stable quasimaps.** Let $\chi(\mathbf{G}) := \mathrm{Hom}(\mathbf{G}, \mathbb{C}^*)$ be the group of characters of a reductive algebraic group $\mathbf{G}$. For $\theta \in \chi(\mathbf{G})$ and a positive rational number ε, the notion of ε-stable quasimaps to the GIT quotient $W/\!\!/_\theta \mathbf{G} = [W^{ss}(\theta)/\mathbf{G}]$ was introduced in [10] provided with the following assumption:

Condition $\bigstar$: The $\mathbf{G}$-action on the semistable locus $W^{ss}(\theta)$ with respect to θ is free.

Note that condition $\bigstar$ guarantees that the stable and semi-stable loci in W for the linearization of the action given by θ coincide.

It will be convenient to extend the notion of stability to a *rational* character θ, while removing ε. This is based on the observation from [10, Remark 7.1.4] that ε-stability with respect to the integral character θ is equivalent to $\frac{\varepsilon}{m}$-stability with respect to $m\theta$, for every positive integer m, and is done as follows. Let

$$\theta \in \chi(\mathbf{G})_{\mathbb{Q}} := \chi(\mathbf{G}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

be a rational character of $\mathbf{G}$. Denote by L_θ the $\mathbb{Q}$-line bundle on $[W/\mathbf{G}]$ associated to the character θ, namely,

$$L_\theta := (W \times \mathbb{C}_{m\theta})^{\otimes 1/m},$$

for any positive integer m making $m\theta$ integral, where $\mathbb{C}_{m\theta}$ stands for the 1-dimensional $\mathbf{G}$-representation space given by the character $m\theta$. Here and in the rest of the paper we identify as usual the $\mathbf{G}$-equivariant

Picard group of W with the Picard group of the quotient stack $[W/\mathbf{G}]$. The unstable closed subscheme $W^{un}(\theta) \subset W$ is defined as $W^{un}(m\theta)$, and the semistable locus is the open subscheme $W^{ss}(\theta) := W \setminus W^{un}(\theta)$. These are independent on the choice of $m \in \mathbb{Z}_{>0}$ with $m\theta \in \chi(\mathbf{G})$. We require that θ satisfies Condition ★ (this makes sense by the above discussion), so that $W/\!\!/_\theta \mathbf{G} = [W^{ss}(\theta)/\mathbf{G}]$.

Definition 2.1. *Let C be a (possibly disconnected) reduced, projective, at worst nodal curve. The* numerical class *of a morphism $f : C \to [W/\mathbf{G}]$ is the homomorphism of abelian groups*

$$\beta_f \in \mathrm{Hom}(\mathrm{Pic}[W/\mathbf{G}], \mathbb{Z})$$

given by

$$\beta_f(L) = \deg f^*(L)$$

for $L \in \mathrm{Pic}([W/\mathbf{G}])$.

Definition 2.2. *Let $(C, \mathbf{x}) := (C, x_1, ..., x_k)$ be a genus g, k-pointed prestable curve over the field $\mathbb{C}$. (Recall this means that C is a reduced, projective, connected, at worst nodal curve of arithmetic genus g, and x_i are distinct nonsingular closed points in C.) A morphism*

$$f : C \longrightarrow [W/\mathbf{G}]$$

is called a k-pointed prestable map of genus g *to* $[W/\mathbf{G}]$.

Definition 2.3. *Let $((C, \mathbf{x}), f)$ be a prestable map to $[W/\mathbf{G}]$.*

- *The* base locus of f with respect to θ *is*

$$f^{-1}([W^{un}(\theta)/\mathbf{G}]) := [W^{un}(\theta)/\mathbf{G}] \times_{[W/\mathbf{G}]} C$$

  with the reduced scheme structure.
- *$((C, \mathbf{x}), f)$ is called a* θ-quasimap to $[W^{ss}(\theta)/\mathbf{G}]$ *if the base locus with respect to θ is purely 0-dimensional.*
- *A θ-quasimap $((C, \mathbf{x}), f)$ is called* θ-prestable *if the base locus is away from all nodes of C.*[1]

By [10, Lemma 3.2.1], a θ-quasimap satisfies

$$\beta_f(L_\theta) \geq 0,$$

with equality if and only if $\beta_f = 0$, if and only if f is a constant map to the GIT quotient $W/\!\!/_\theta \mathbf{G} = [W^{ss}(\theta)/\mathbf{G}]$.

[1]The definition of prestability given here differs slightly from that in [10, Definition 3.1.2], as we now allow base-points to occur at the markings of a prestable quasimap. The stability condition (2) in Definition 2.6 below implies that there are no base-points at markings for *stable* quasimaps. This choice of definitions is more natural from the perspective of the weighted case introduced in §2.2.

Definition 2.4. *Let $((C, \mathbf{x}), f)$ be a θ-prestable quasimap to $[W^{ss}(\theta)/\mathbf{G}]$. The θ*-length *$\ell_\theta(p)$ of f at a smooth closed point p of C is defined as follows: Choose $\varepsilon' \in \mathbb{Q}_{>0}$ such that $\theta' = \frac{1}{\varepsilon'}\theta \in \chi(\mathbf{G})$ is an integral character. Then*

$$\ell_\theta(p) := \varepsilon' \ell_{\theta'}(p),$$

where $\ell_{\theta'}(p)$ is the length defined in [10, Definition 7.1.1].

Remark 2.5. The following properties are immediate to check from the above definition:

(1) $\ell_\theta(p)$ is a well-defined rational number (i.e., it does not depend on the choice of ε' and θ'). If $\lambda \in \mathbb{Q}_{>0}$, then $\ell_{\lambda\theta}(p) = \lambda \ell_\theta(p)$.

(2) For every nonsingular point $p \in C$,

$$0 \leq \ell_\theta(p) \leq \beta_f(L_\theta)$$

and $\ell_\theta(p) > 0$ if and only if p is in the base locus of f.

(3) Suppose that W is a product $W_1 \times W_2$ of two affine varieties W_i with component-wise $\mathbf{G} := \mathbf{G}_1 \times \mathbf{G}_2$-action such that Condition $\bigstar$ holds for each pair (W_i, θ_i). Let θ_i be the character of the reductive group $\mathbf{G}_i$ induced from the character θ of $\mathbf{G}$, so that $\theta = \theta_1 \oplus \theta_2$. For a prestable map

$$f = (f_1, f_2) : C \to [W/\mathbf{G}] = [W_1/\mathbf{G}_1] \times_{\mathrm{Spec}\,\mathbb{C}} [W_2/\mathbf{G}_2]$$

and a smooth point $p \in C$,

$$\ell_\theta(p) = \ell_{\theta_1}(p) + \ell_{\theta_2}(p).$$

This follows from the Künneth formula.

Definition 2.6. *A θ-prestable quasimap $((C, \mathbf{x}), f)$ is θ*-stable *if:*

(1) *$\omega_C(\sum x_i) \otimes f^* L_\theta$ is ample and*

(2) *for every smooth point $p \in C$,*

$$\ell_\theta(p) + \sum_i \delta_{x_i, p} \leq 1$$

where $\delta_{x_i,p} := 1$ if $x_i = p$; $\delta_{x_i,p} := 0$ if $x_i \neq p$.

Note that the stability condition (2) in Definition 2.6 requires that $\ell_\theta(x_i) = 0$ for each marking x_i. By Remark 2.5(2), this says that the base locus of a θ-stable quasimap is away from the markings of C.

Proposition 2.7. *Let $\theta = \varepsilon'\theta'$ with $\varepsilon' \in \mathbb{Q}_{>0}$ and θ' integral. Then*

(*i*) *A prestable map $((C, \mathbf{x}), f)$ to $[W/\mathbf{G}]$ is θ-stable if and only if it is a ε'-stable quasimap to $W/\!\!/_{\theta'}\mathbf{G}$, as defined in* [10, Definition 7.1.3].

(ii) A prestable map $((C,\mathbf{x}),f)$ *to* $[W/\mathbf{G}]$ *with* $\beta_f(L_\theta)\leq 1$ *is* θ*-stable if and only if it is a stable quasimap to* $W/\!\!/_{\theta'}\mathbf{G}$*, as defined in* [10, Definition 3.1.2] *(or a* (0+)*-stable quasimap to* $W/\!\!/_{\theta'}\mathbf{G}$*, in the terminology of* [7, Remark 2.4.7(2)]*).*

(iii) Let θ_0 *be the minimal integral character in the half ray* $\mathbb{Q}_{>0}\theta$*. We write* $\theta_1>\theta_2$ *if* $\theta_1=\lambda_1\theta_0$ *and* $\theta_2=\lambda_2\theta_0$ *with two positive rational numbers* $\lambda_1>\lambda_2$*.*

If $\theta>\theta_0$ *and* $(g,k)\neq(0,0)$ *(*$\theta>2\theta_0$ *when* $(g,k)=(0,0)$*), a prestable map* $((C,\mathbf{x}),f)$ *to* $[W/\mathbf{G}]$ *is* θ*-stable if and only if it is a stable map to the quasi-projective scheme* $W/\!\!/_{\theta'}\mathbf{G}$*.*

Proof. Left to the reader, as all statements follow easily from the definitions. □

Definition 2.8. *An element* $\beta\in\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}([W/\mathbf{G}],\mathbb{Z})$ *is called* θ-effective *(or equivalently* $L_{\theta'}$*-effective as in* [10, Definition 3.2.2]*) if it can be realized as a finite sum of classes of* θ*-quasimaps.*

The subset $\mathrm{Eff}(W,G,\theta)\subset\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}([W/\mathbf{G}],\mathbb{Z})$ of θ-effective classes is a semigroup with no nontrivial invertible elements, i.e., $\beta_1+\beta_2=0$ for $\beta_i\in\mathrm{Eff}(W,G,\theta)$ implies that $\beta_1=\beta_2=0$ (see [10, Lemma 3.2.1]).

For a θ-effective class β, we denote by $Q^\theta_{g,k}([W/\mathbf{G}],\beta)$ the moduli stack of genus g, k-pointed θ-stable quasimaps to $[W/\mathbf{G}]$ with numerical class β. By Proposition 2.7(i),

$$Q^\theta_{g,k}([W/\mathbf{G}],\beta)=Q^{\varepsilon'}_{g,k}(W/\!\!/_{\theta'}\mathbf{G},\beta), \tag{2.1.1}$$

where the right-hand side is the stack from [10, Theorem 7.1.6]. Hence $Q^\theta_{g,k}([W/\mathbf{G}],\beta)$ is a DM-stack, proper over the affine quotient

$$W/_{\mathrm{aff}}\mathbf{G}:=\mathrm{Spec}(A(W)^{\mathbf{G}}),$$

where $A(W)$ denotes the affine coordinate ring of W. These moduli stacks carry canonical perfect obstruction theories (see [10, §4.4-4.5]).

Definition 2.9. *A prestable map* $((C,\mathbf{x}),f)$ *to* $[W/\mathbf{G}]$ *which is* $\lambda\theta$*-stable for every* $0<\lambda<<1$ *is called* $(0+)\cdot\theta$-stable*. This notion is equivalent to the notion of stable quasimaps with respect to* θ' *defined in* [10, Definition 3.1.2]*, where* θ' *is any integral character in the half ray* $\mathbb{Q}_{>0}\theta$*. See also* [7, Remark 2.4.7(2)]*, where the terminology* (0+)*-stable quasimaps to* $W/\!\!/_{\theta'}\mathbf{G}$ *was used for the same notion.*

Therefore we define the corresponding moduli stacks by

$$Q^{(0+)\cdot\theta}_{g,k}([W/\mathbf{G}],\beta):=Q^{0+}_{g,k}(W/\!\!/_{\theta'}\mathbf{G},\beta), \tag{2.1.2}$$

where for the right-hand side we used the notation from [7, Remark 2.4.7(2)]. They are also DM-stacks, proper over the affine quotient, carrying canonical perfect obstruction theories.

We discuss next θ-stability for the quasimap graph spaces of [10, §7.2] and [7, §2.6].

Let $N \geq 1$ be an integer and consider the standard scaling action of $\mathbb{C}^*$ on $\mathbb{C}^N$. For $n \in \mathbb{Z}$ we have the character

$$n\mathrm{id} : \mathbb{C}^* \longrightarrow \mathbb{C}^*, \quad t \mapsto t^n.$$

There are identifications

$$\mathbb{Z} \xrightarrow{\sim} \chi(\mathbb{C}^*) \xrightarrow{\sim} \mathrm{Pic}([\mathbb{C}^N/\mathbb{C}^*], \quad n \mapsto n\mathrm{id} \mapsto L_{n\mathrm{id}}.$$

For each $\beta \in \mathrm{Hom}(\mathrm{Pic}([W/\mathbf{G}]), \mathbb{Z})$, define an abelian group homomorphism $(\beta, 1) \in \mathrm{Hom}(\mathrm{Pic}([W/\mathbf{G}] \times [\mathbb{C}^N/\mathbb{C}^*]), \mathbb{Z})$ by

$$(\beta, 1)(L \boxtimes L_{n\mathrm{id}}) = \beta(L) + n.$$

Now we define the *θ-stable quasimap graph space*:

$$QG^{\theta}_{g,k,\beta}([W/\mathbf{G}]) := Q^{\theta \oplus 3\mathrm{id}}_{g,k}([W \times \mathbb{C}^2/\mathbf{G} \times \mathbb{C}^*], (\beta, 1)), \tag{2.1.3}$$

where $\theta \oplus 3\mathrm{id}$ is a rational character of $\mathbf{G} \times \mathbb{C}^*$. As before, we see that

$$QG^{\theta}_{g,k,\beta}([W/\mathbf{G}]) = QG^{\varepsilon'}_{g,k,\beta}(W/\!\!/_{\theta'}\mathbf{G}), \tag{2.1.4}$$

where the right-hand side is the graph space of ε'-stable quasimaps to the GIT quotient (in the notation from [7, §2.6]).

Finally, we have the graph spaces for the $(0+) \cdot \theta$-stability condition:

$$QG^{(0+)\cdot\theta}_{g,k,\beta}([W/\mathbf{G}]) := QG^{0+}_{g,k,\beta}(W/\!\!/_{\theta'}\mathbf{G}). \tag{2.1.5}$$

Again, the graph spaces (2.1.3) and (2.1.5) are DM-stacks, proper over the affine quotient, and carry canonical perfect obstruction theories.

2.2. **Weighted stable quasimaps.** In this section, we introduce the weighted pointed stable quasimaps. The moduli spaces of weighted pointed stable maps to a (quasi)projective target are constructed and studied in [1, 2, 17]. Recently, in [19], Janda considered the moduli space of weighted pointed stable quotients and its applications. Also recently, in [20], Jinzenji and Shimizu studied a graph space-type quasimap compactification of the moduli space of maps from $\mathbb{P}^1$ to $\mathbb{P}^n$ with some weighted markings and its applications to generalized mirror maps.

Let

$$(\theta, \epsilon) := (\theta, \varepsilon_1, ..., \varepsilon_k) \in \chi(\mathbf{G})_{\mathbb{Q}} \times (\mathbb{Q}_{>0})^k$$

such that θ satisfies Condition ★ and $\varepsilon_i \leq 1$, $i = 1, ..., k$.

Definition 2.10. *A pair* $((C, x_1, ..., x_k), f)$ *is called a* (θ, ϵ)-stable quasimap with weighted markings *and numerical class* β *if:*

(1) (ϵ-weighted prestable map to $[W/\mathbf{G}]$)
   (a) *C is a genus g, prestable curve over the field $\mathbb{C}$.*
   (b) *x_i are smooth points on C (not necessarily pairwise distinct), with*
$$\sum_i \varepsilon_i \delta_{x_i,p} \leq 1$$
   for every smooth point p of C.
   (c) *f is a morphism from C to $[W/\mathbf{G}]$.*

(2) (θ-quasimap) $f^{-1}([W^{un}(\theta)/\mathbf{G}])$ *is pure* 0*-dimensional.*

(3) (θ-prestability) $f^{-1}(W/\!\!/_\theta\mathbf{G})$ *contains all nodes of C.*

(4) ((θ, ϵ)-stability)
   (a) *The $\mathbb{Q}$-line bundle*
$$\omega_C(\sum_{j=1}^{k} \varepsilon_i x_i) \otimes f^* L_\theta$$
   is ample.
   (b) *For every smooth point $p \in C$,*
$$\ell_\theta(p) + \sum_i \varepsilon_i \delta_{x_i,p} \leq 1.$$

(5) (numerical class β) $\beta_f = \beta$.

By treating each marking x_i as an effective divisor of C, there is a natural correspondence

$$\left\{ \begin{array}{r} (f : C \to [W/\mathbf{G}]),\ \text{together with ordered smooth points} \\ x_i \in C, i = 1, ..., k :\ \text{class } \beta \end{array} \right\}$$
$$\leftrightarrow \left\{ \begin{array}{r} (\tilde{f} := (f, \pi_1, ..., \pi_k) : C \to [W/\mathbf{G}] \times [\mathbb{C}/\mathbb{C}^*]^k) : \\ \pi_i \text{ are id-prestable quasimaps to } [\mathbb{C}/\mathbb{C}^*],\ \text{class } (\beta, 1, ..., 1) \end{array} \right\}.$$

Consider the rational character

$$\boldsymbol{\theta} := \theta \oplus \underbrace{\varepsilon_1 \mathrm{id} \oplus \cdots \oplus \varepsilon_k \mathrm{id}}_{k} \in \chi(\mathbf{G} \times (\mathbb{C}^*)^k)_{\mathbb{Q}}.$$

Then $\tilde{f}^*(L_{\boldsymbol{\theta}}) = f^*(L_\theta) \otimes \mathcal{O}_C(\sum \varepsilon_i x_i)$ and $\ell_{\boldsymbol{\theta}}(p) = \ell_\theta(p) + \sum \varepsilon_i \delta_{x_i,p}$. Therefore, the (θ, ϵ)-stability of $((C, x_1, ..., x_k), f)$ from Definition 2.10 translates via the above correspondence into $\boldsymbol{\theta}$-stability of $\tilde{f}$, and so the moduli stack of (θ, ϵ)-stable quasimaps of type (g, β) is identified with

$$Q^{\boldsymbol{\theta}}_{g,0}([W/\mathbf{G}] \times [\mathbb{C}/\mathbb{C}^*]^k), (\beta, 1, ..., 1)).$$

By (2.1.1), it is a DM stack, proper over $W/\!\!/_{\text{aff}}\mathbf{G}$, with a canonical perfect obstruction theory. Note that

$$2g-2+\sum_{i=1}^{k}\varepsilon_i+\beta(L_\theta)>0$$

is a necessary condition for the moduli stack to be non-empty.

In the rest of the paper we will be interested in a particular case. Namely, replace k by $m+k$ and then let $\varepsilon_i=1$ for all $i\le m$ and $\varepsilon_{m+j}=\varepsilon$, with ε a fixed positive rational number for $j=1,...,k$. We denote the ordered markings by $x_1,...,x_m,y_1,...,y_k$. Hence, if

$$((C,\mathbf{x}:=(x_1,...,x_m),\mathbf{y}:=(y_1,...,y_k)),f)$$

is $(\theta,\boldsymbol{\epsilon})$-stable, then $(C,\mathbf{x})$ is a m-pointed prestable curve and x_i are not base points of f. In addition, while the points y_j are allowed to coincide, no point y_j may coincide with any of the x_i's. In this case, we also simply say that it is (θ,ε)-stable. Denote by

$$Q^{\theta,\varepsilon}_{g,m|k}([W/\mathbf{G}],\beta)$$

the moduli space of (θ,ε)-stable maps to $[W/\mathbf{G}]$ of type $(g,m|k,\beta)$.

If $((C,\mathbf{x},\mathbf{y}),f)$ is $(\lambda\theta_0,\varepsilon)$-stable for every sufficiently small rational number $0<\lambda<<1$ (respectively, every sufficiently large rational number λ, every $0<\varepsilon<<1$, ...), then we say that it is $((0+)\cdot\theta_0,\varepsilon)$-stable (respectively, $(\infty\cdot\theta_0,\varepsilon)$-stable, $(\theta,0+)$-stable, ...). Thus, from now on we consider the following extended cases

$$(\theta,\varepsilon)\in(\chi(\mathbf{G})_{\mathbb{Q}}\cup\{(0+)\cdot\theta_0,\infty\cdot\theta_0\})\times(((0,1]\cap\mathbb{Q})\cup\{0+\})).$$

We treat 0+ as an infinitesimally small *positive* rational number.

Remark 2.11. When $[W/\mathbf{G}]=[\mathbb{C}^{n+1}/\mathbb{C}^*]$ with $W/\!\!/\mathbf{G}=\mathbb{P}^n$, it is worth to note that the genus 1 moduli space $Q^{\text{id},0+}_{1,0|k}([\mathbb{C}^{n+1}/\mathbb{C}^*],\beta)$ is a smooth DM-stack over $\mathbb{C}$ since the obstruction vanishes (see [22]).

2.3. **Evaluation maps.** There are evaluation maps at y_j, $j=1,...,k$,

$$\hat{ev}_j: Q^{\theta,\varepsilon}_{g,m|k}([W/\mathbf{G}],\beta)\to[W/\mathbf{G}]$$

as well as the usual evaluation maps ev_i at x_i, $i=1,...,m$,

$$\begin{array}{ccc} Q^{\theta,\varepsilon}_{g,m|k}([W/\mathbf{G}],\beta) & \xrightarrow{ev_i} & W/\!\!/_\theta\mathbf{G} \\ & \searrow^{\text{proper}} & \downarrow{\scriptstyle\text{proper}} \\ & & W/\!\!/_{\text{aff}}\mathbf{G} \end{array}$$

compatible with canonical maps to $W/_{\mathrm{aff}}\mathbf{G}$. The evaluation maps ev_i, $i \in [m] := \{1, ..., m\}$ are proper, so the push-forward of homology or Chow classes on $Q^{\theta,\varepsilon}_{g,m|k}([W/\mathbf{G}], \beta)$ is well-defined.

3. The big $\mathbb{J}$-functions

3.1. **The Novikov ring.** Let an algebraic torus $\mathbf{T}$ act on W, commuting with the $\mathbf{G}$-action. Recall we allow the case when $\mathbf{T}$ is the trivial group. Denote

$$H^*_{\mathbf{T}}(\mathrm{Spec}(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}[\lambda_1, ..., \lambda_r]$$

the $\mathbf{T}$-equivariant cohomology of a point $\mathrm{Spec}(\mathbb{C})$, where r is the rank of $\mathbf{T}$. Define the Novikov ring

$$\Lambda := \{ \sum_{\beta \in \mathrm{Eff}(W,\mathbf{G},\theta)} a_\beta q^\beta : a_\beta \in \mathbb{Q}\},$$

the q-adic completion of the semigroup ring $\mathbb{Q}[\mathrm{Eff}(W, \mathbf{G}, \theta)]$, and set

$$\Lambda_{\mathbf{T}} := \Lambda \otimes_{\mathbb{Q}} \mathbb{Q}[\lambda_1, ..., \lambda_r],$$
$$\Lambda_{\mathbf{T},\mathrm{loc}} := \Lambda_{\mathbf{T}} \otimes \mathbb{Q}(\lambda_1, ..., \lambda_r).$$

3.2. **Weighted graph spaces.** As in (2.1.3), we define the (θ, ε)-*stable quasimap graph space* as follows:

$$QG^{\theta,\varepsilon}_{g,m|k,\beta}([W/\mathbf{G}]) := Q^{\theta\oplus 3\mathrm{id},\varepsilon}_{g,m|k}([W \times \mathbb{C}^2/\mathbf{G} \times \mathbb{C}^*], (\beta, 1)).$$

A $\mathbb{C}$-point of the graph space is described by data

$$((C, \mathbf{x}, \mathbf{y}), (f, \varphi) : C \longrightarrow [W/\mathbf{G}] \times [\mathbb{C}^2/\mathbb{C}^*]).$$

Since $\ell_{3\mathrm{id}}(p)$ equals either 0 or 3 for every smooth point $p \in C$, stability implies that φ is a regular map to $\mathbb{P}^1 = \mathbb{C}^2/\!\!/_{\mathrm{id}}\mathbb{C}^*$, of class 1. Hence the domain curve C has a distinguished irreducible component C_0 canonically isomorphic to $\mathbb{P}^1$ via φ. The "standard" $\mathbb{C}^*$-action,

$$t \cdot [\xi_0, \xi_1] = [t\xi_0, \xi_1], \text{ for } t \in \mathbb{C}^*, [\xi_0, \xi_1] \in \mathbb{P}^1,$$

induces a $\mathbb{C}^*$-action on the graph space. With this convention, the $\mathbb{C}^*$-equivariant first Chern class of the tangent line $T_0\mathbb{P}^1$ at $0 \in \mathbb{P}^1$ is $c_1^{\mathbb{C}^*}(T_0\mathbb{P}^1) = z$, where z denotes the equivariant parameter, i.e., $H^*_{\mathbb{C}^*}(\mathrm{Spec}(\mathbb{C})) = \mathbb{Q}[z]$.

There are $\mathbf{T} \times \mathbb{C}^*$-equivariant evaluation morphisms

$$\hat{\widetilde{ev}}_j : QG^{\theta,\varepsilon}_{g,m|k,\beta}([W/\mathbf{G}]) \to [W/\mathbf{G}] \times \mathbb{P}^1, \qquad j = 1, \dots, k,$$
$$\widetilde{ev}_i : QG^{\theta,\varepsilon}_{g,m|k,\beta}([W/\mathbf{G}]) \to W/\!\!/_\theta\mathbf{G} \times \mathbb{P}^1, \qquad i = 1, \dots, m,$$

and

$$\hat{ev}_j := pr_1 \circ \hat{\widetilde{ev}}_j : QG^{\theta,\varepsilon}_{g,m|k,\beta}([W/\mathbf{G}]) \to [W/\mathbf{G}], \qquad j = 1, \dots, k,$$
$$ev_i := pr_1 \circ \widetilde{ev}_i : QG^{\theta,\varepsilon}_{g,m|k,\beta}([W/\mathbf{G}]) \to W/\!\!/_\theta \mathbf{G}, \qquad i = 1, \dots, m,$$

where pr_1 is the projection to the first factor.

Since to give a morphism $f : C \to [W/\mathbf{G}]$ amounts to giving a principal $\mathbf{G}$-bundle P on C and a section u of $P \times_{\mathbf{G}} W$, there is a natural morphism $C \to E\mathbf{G} \times_{\mathbf{G}} W$ and hence a pull-back homomorphism

$$f^* : H^*_{\mathbf{G}}(W) \to H^*(C).$$

Now apply this to the universal curve over the moduli space, with its universal morphism to $[W/\mathbf{G}]$. The evaluation maps are the compositions of the universal morphism with the sections of the universal curve giving the markings and are $\mathbf{T} \times \mathbb{C}^*$-equivariant. We obtain in this way the pull-back homomorphism

$$\hat{ev}_j^* : H^*_{\mathbf{G}\times\mathbf{T}}(W, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[z] \to H^*_{\mathbf{T}\times\mathbb{C}^*}(QG^{\theta,\varepsilon}_{g,m|k,\beta}([W/\mathbf{G}]), \mathbb{Q})$$

associated to the evaluation map $\hat{ev}_j$.

We identify as usual $H^*_{\mathbf{T}}([W/\mathbf{G}], \mathbb{Q}) := H^*_{\mathbf{G}\times\mathbf{T}}(W, \mathbb{Q})$.

Now fix (θ, ε) (including the cases $\theta = (0+) \cdot \theta_0$ and $\varepsilon = 0+$) and consider the graph spaces $QG^{\theta,\varepsilon}_{0,0|k,\beta}([W/\mathbf{G}])$. The description of the fixed loci for the $\mathbb{C}^*$-action is parallel to the one given in [7, §4.1] for the unweighted case. In particular, we have the part $F_{k,\beta}$ of the $\mathbb{C}^*$-fixed locus for which the markings and the entire class β are over $0 \in \mathbb{P}^1$. It comes with a natural *proper* evaluation map $ev_\bullet$ at the generic point of $\mathbb{P}^1$:

$$ev_\bullet : F_{k,\beta} \to W/\!\!/\mathbf{G}.$$

When $k\varepsilon + \beta(L_\theta) > 1$, we have the identification

$$F_{k,\beta} \cong Q^{\theta,\varepsilon}_{0,1|k}([W/\mathbf{G}], \beta),$$

with $ev_\bullet = ev_1$, the evaluation map at the weight 1 marking.

On the other hand, when $k\varepsilon + \beta(L_\theta) \le 1$, then

$$F_{k,\beta} \cong F_\beta \times 0^k \subset F_\beta \times (\mathbb{P}^1)^k,$$

with F_β the $\mathbb{C}^*$-fixed locus in $QG^{(0+)\cdot\theta}_{0,0,\beta}([W/\mathbf{G}])$ for which the class β is concentrated over $0 \in \mathbb{P}^1$. This F_β parametrizes quasimaps of class β

$$f : \mathbb{P}^1 \longrightarrow [W/\mathbf{G}]$$

with a base-point of length $\beta(L_\theta)$ at $0 \in \mathbb{P}^1$. The restriction of f to $\mathbb{P}^1 \setminus \{0\}$ is a constant map to $W/\!\!/_\theta \mathbf{G}$ and this defines the evaluation map $ev_\bullet$.

As in [6, 10, 7], we define the big $\mathbb{J}$-function as the generating function for the push-forward via $ev_\bullet$ of localization residue contributions of $F_{k,\beta}$:

Definition 3.1. *For $\mathbf{t} \in H^*_{\mathbf{T}}([W/\mathbf{G}], \mathbb{Q}) \subset H^*_{\mathbf{T}}([W/\mathbf{G}], \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[z]$, let*

$$\begin{aligned} \mathrm{Res}_{F_{k,\beta}}(\mathbf{t}^k) &:= (\iota_\beta^*(\prod_{i=1}^{k} \hat{ev}_i^*(\mathbf{t}))) \cap \mathrm{Res}_{F_{k,\beta}}[QG^{\theta,\varepsilon}_{0,0|k,\beta}([W/\mathbf{G}])]^{\mathrm{vir}} \\ &:= \frac{(\iota_\beta^*(\prod_{i=1}^{k} \hat{ev}_i^*(\mathbf{t}))) \cap [F_{k,\beta}]^{\mathrm{vir}}}{\mathrm{e}^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_{k,\beta}})}, \end{aligned}$$

where $\iota_\beta : F_\beta \hookrightarrow QG^{\theta,\varepsilon}_{0,0|k,\beta}([W/\mathbf{G}])$ is the inclusion, $N^{\mathrm{vir}}_{F_{k,\beta}}$ is the virtual normal bundle and $\mathrm{e}^{\mathbb{C}^*}$ denotes the equivariant Euler class.

The big $\mathbb{J}$-function for the (θ, ε)-stability condition is

$$\mathbb{J}^{\theta,\varepsilon}(q, \mathbf{t}, z) := \sum_{\beta \in \mathrm{Eff}(W,\mathbf{G},\theta)} \sum_{k \geq 0} \frac{q^\beta}{k!} (ev_\bullet)_* \mathrm{Res}_{F_{k,\beta}}(\mathbf{t}^k) \tag{3.2.1}$$

as a formal function in $\mathbf{t}$.

Usually we will only be concerned with the restriction of $\mathbf{t}$ to a finite dimensional subspace of $H^*_{\mathbf{T}}([W/\mathbf{G}], \mathbb{Q})$ as follows. Let

$$\kappa : H^*_{\mathbf{T}}([W/\mathbf{G}], \mathbb{Q}) \to H^*_{\mathbf{T}}(W/\!\!/_\theta \mathbf{G}, \mathbb{Q})$$

denote the Kirwan map (surjective, by [21]) induced from the open immersion $W/\!\!/_\theta \mathbf{G} = [W^{ss}(\theta)/\mathbf{G}] \subset [W/\mathbf{G}]$.

Fix a homogeneous basis $\{\gamma_i\}_i$ of $H^*_{\mathbf{T}}(W/\!\!/\mathbf{G})$ and choose homogeneous lifts $\tilde{\gamma}_i \in H^*_{\mathbf{T}}([W/\mathbf{G}], \mathbb{Q})$ with $\kappa(\tilde{\gamma}_i) = \gamma_i$. After restricting to

$$\mathbf{t} := \sum_i t_i \tilde{\gamma}_i,$$

the big $\mathbb{J}$-function (3.2.1) is a formal function in the finitely many variables $\{t_i\}$.

We remark that $\hat{ev}_i^*(\mathbf{t})$ is a class in $H^*_{\mathbf{T}\times\mathbb{C}^*}(QG^{\theta,\varepsilon}_{0,0|k,\beta}([W/\mathbf{G}]), \mathbb{Q})$.

Since

$$QG^{\theta,\varepsilon}_{0,0|k,\beta=0}([W/\mathbf{G}]) = W/\!\!/_\theta \mathbf{G} \times (\mathbb{P}^1)^k \supset F_{k,0} = W/\!\!/_\theta \mathbf{G} \times 0^k,$$

we conclude that

$$\mathbb{J}^{\theta,\varepsilon}(\mathbf{t}, z) = e^{\kappa(\mathbf{t})/z} + O(q). \tag{3.2.2}$$

From now on, unless otherwise stated, *assume that the $\mathbf{T}$-fixed locus $(W/_{\mathrm{aff}}\mathbf{G})^{\mathbf{T}}$ is proper over $\mathbb{C}$ (i.e., a finite set of points).* This implies that the $\mathbf{T}$-fixed loci in $W/\!\!/_\theta \mathbf{G}$, as well as the $\mathbf{T}$-fixed loci in all moduli stacks of (θ, ε)-stable quasimaps are also proper.

3.3. **E-Twisting.** Let E be a finite dimensional $\mathbf{T}\times\mathbf{G}$-representation space. Then twisting by a $\mathbf{T}$-equivariant vector bundle

$$\underline{E} := W \times_{\mathbf{G}} E$$

on $[W/\mathbf{G}]$ can be inserted by the replacements

$$[Q^{\theta,\varepsilon}_{0,m|k}([W/\mathbf{G}],\beta)]^{vir} \mapsto \mathrm{e}^{\mathbf{T}}(\pi_* f^* \underline{E}) \cap [Q^{\theta,\varepsilon}_{0,m|k}([W/\mathbf{G}],\beta)]^{vir},$$
$$[QG^{\theta,\varepsilon}_{0,m|k,\beta}([W/\mathbf{G}])]^{vir} \mapsto \mathrm{e}^{\mathbf{T}}(\pi_* f^* \underline{E}) \cap [QG^{\theta,\varepsilon}_{0,m|k,\beta}([W/\mathbf{G}])]^{vir}$$

as in [7, §7.2.1] assuming that

$$R^1\pi_* f^* \underline{E} = 0 \quad \text{for all } \beta \in \mathrm{Eff}(W,\mathbf{G},\theta). \tag{3.3.1}$$

Here π is the projection from the universal curve $\mathcal{C}$, $f : \mathcal{C} \longrightarrow [W/\mathbf{G}]$ is the universal map to the quotient stack, and $\mathrm{e}^{\mathbf{T}}$ is the equivariant Euler class. Note that if $\mathcal{P}$ denotes the universal principal $\mathbf{G}$-bundle on $\mathcal{C}$, then $f^*\underline{E} = \mathcal{P} \times_{\mathbf{G}} E$.

Now we can define $\tilde{\mathbb{J}}^{\theta,\varepsilon,E}$ exactly parallel to [7, §7.2.1]:

$$\tilde{\mathbb{J}}^{\theta,\varepsilon,E}(q,\mathbf{t},z) = \left(\mathbb{1} + \frac{\kappa(\mathbf{t})}{z}\right)\mathrm{e}^{\mathbf{T}}(\underline{E}|_{W/\!\!/\mathbf{G}}) + \sum_{(k,\beta)\neq(0,0),(1,0)} \frac{q^\beta}{k!}\times$$
$$\times\, (ev_\bullet)_*\left(\iota^*_\beta(\prod_{i=1}^k \hat{ev}^*_i(\mathbf{t})) \cap \mathrm{Res}_{F_{k,\beta}}(\mathrm{e}^{\mathbf{T}}(\pi_* f^*\underline{E}) \cap [QG^{\theta,\varepsilon}_{0,0|k,\beta}([W/\mathbf{G}])]^{\mathrm{vir}})\right).$$

3.4. **Results.**

Conjecture 3.2. *The function $\tilde{\mathbb{J}}^{\theta,\varepsilon,E}$ is on the Lagrangian cone encoding the genus 0, $\mathbf{T}$-equivariant, $\underline{E}|_{W/\!\!/\mathbf{G}}$-twisted Gromov-Witten theory of $W/\!\!/_\theta\mathbf{G}$ with the Novikov ring $\Lambda_{\mathbf{T}}$ (see* [13, 16] *for the definition of the Lagrangian cone).*

Theorem 3.3. *If the $\mathbf{T}$-action on $W/\!\!/_\theta\mathbf{G}$ has only isolated fixed points and only isolated 1-dimensional orbits, Conjecture 3.2 holds true.*

4. Proof of Theorem 3.3

To keep the presentation simple, we drop the E-twisting. However, an identical proof works in the twisted case as well.

Let $\{\gamma_i := \kappa(\tilde{\gamma}_i)\}$ be a basis of

$$H^*_{\mathbf{T},\mathrm{loc}}(W/\!\!/_\theta\mathbf{G}) := H^*_{\mathbf{T}}(W/\!\!/_\theta\mathbf{G},\mathbb{Q}) \otimes_{\mathbb{Q}[\lambda_1,\dots,\lambda_r]} \mathbb{Q}(\lambda_1,\dots,\lambda_r)$$

and $\{\gamma^i\}$ be the dual basis with respect to the $\mathbf{T}$-equivariant Poincaré pairing $\langle\ ,\ \rangle$ of $W/\!\!/_\theta\mathbf{G}$.

4.1. **The $\mathbb{S}$-operator.** For $\sigma_i \in H^*_{\mathbf{T},\mathrm{loc}}(W/\!\!/_\theta \mathbf{G})$ and $\delta_j \in H^*_{\mathbf{T}}([W/\mathbf{G}], \mathbb{Q})$, denote

$$\langle \sigma_1\psi_1^{a_1}, ..., \sigma_m\psi_m^{a_m}; \delta_1, ..., \delta_k\rangle^{\theta,\varepsilon}_{g,m|k,\beta} := \int_{[Q^{\theta,\varepsilon}_{g,m|k}(W/\!\!/\mathbf{G},\beta)]^{\mathrm{vir}}} \prod_i ev_i^*(\sigma_i)\psi_i^{a_i} \prod_j \hat{ev}_j^*(\delta_j),$$

where ψ_i is psi-class associated to the i^{th}-marking of weight 1. In the case $W/_{\mathrm{aff}}\mathbf{G}$ is not a single point, so that $W/\!\!/\mathbf{G}$ is only quasi-projective, the integral is understood as usual via the virtual localization formula.

Define for a formal $\mathbf{t} = \sum t_i \tilde{\gamma}_i$ in $H^*_{\mathbf{T}}([W/\mathbf{G}], \mathbb{Q})$

$$\langle\!\langle \sigma_1\psi_1^{a_1}, ..., \sigma_m\psi_m^{a_m}\rangle\!\rangle^{\theta,\varepsilon}_{g,m,\beta} := \sum_{k\geq 0} \frac{1}{k!}\langle \sigma_1\psi_1^{a_1}, ..., \sigma_m\psi_m^{a_m}; \mathbf{t}, ..., \mathbf{t}\rangle^{\theta,\varepsilon}_{g,m|k,\beta},$$

$$\langle\!\langle \sigma_1\psi_1^{a_1}, ..., \sigma_m\psi_m^{a_m}\rangle\!\rangle^{\theta,\varepsilon}_{g,m} := \sum_{\beta} q^\beta \langle\!\langle \sigma_1\psi_1^{a_1}, ..., \sigma_m\psi_m^{a_m}\rangle\!\rangle^{\theta,\varepsilon}_{g,m,\beta}.$$

Remark 4.1. Let $\mathbf{T}$ be the trivial group. Then without the assumption that $W/_{\mathrm{aff}}\mathbf{G}$ is a point, we may regard the above invariants as taking values in Borel-Moore homology $H^{\mathrm{BM}}_*(W/_{\mathrm{aff}}\mathbf{G}, \Lambda_{\mathrm{nov}})$ using the canonical proper morphism $Q^{\theta,\varepsilon}_{g,m|k}(W/\!\!/\mathbf{G},\beta) \to W/_{\mathrm{aff}}\mathbf{G}$.

We define next the $\mathbb{S}$-operator: for $\gamma \in H^*_{\mathbf{T},\mathrm{loc}}(W/\!\!/_\theta\mathbf{G}, \Lambda)$,

$$\mathbb{S}^{\theta,\varepsilon}_{\mathbf{t}}(z)(\gamma) := \sum_i \gamma_i \langle\!\langle \frac{\gamma^i}{z-\psi}, \gamma\rangle\!\rangle^{\theta,\varepsilon}_{0,2} = \gamma + O(1/z). \tag{4.1.1}$$

Let $\overline{M}_{0,2|\varepsilon\cdot k}$ be the Hassett moduli space of $(1, 1, \varepsilon, ..., \varepsilon)$-weighted stable pointed curves. By [17] there is a natural birational contraction

$$\overline{M}_{0,2+k} \to \overline{M}_{0,2|\varepsilon\cdot k}.$$

From this and the identification

$$Q^{\theta,\varepsilon}_{0,2|k}(W/\!\!/\mathbf{G}, 0) = \overline{M}_{0,2|\varepsilon\cdot k} \times W/\!\!/\mathbf{G}$$

of the moduli spaces with class $\beta = 0$ one obtains that the $\mathbb{S}$-operator has the asymptotic expansion in q

$$\mathbb{S}^{\theta,\varepsilon}_{\mathbf{t}}(z)(\gamma) = e^{\kappa(\mathbf{t})/z}\gamma + O(q). \tag{4.1.2}$$

Let p_0 and p_∞ be $\mathbb{C}^*$-equivariant cohomology classes of $\mathbb{P}^1$ defined by their restriction at the fixed points:

$$p_0|_0 = z,\ p_0|_\infty = 0,\ p_\infty|_0 = 0,\ p_\infty|_\infty = -z.$$

Consider the graph space double bracket

$$\langle\langle \sigma_1 \otimes p_0, \sigma_2 \otimes p_\infty \rangle\rangle_{0,2}^{QG^{\theta,\varepsilon}} :=$$
$$\sum_{k,\beta} \frac{q^\beta}{k!} \int_{[QG^{\theta,\varepsilon}_{0,2|k,\beta}([W/\mathbf{G}])]^{\mathrm{vir}}} \widetilde{ev}_1^*(\sigma_1 \otimes p_0) \widetilde{ev}_2^*(\sigma_2 \otimes p_\infty) \prod_{j=1}^{k} \hat{ev}_j^*(\mathbf{t}) =$$
$$\langle \sigma_1, \sigma_2 \rangle + O(q).$$

Virtual $\mathbb{C}^*$-localization gives the factorization

$$\begin{aligned} \langle\langle \sigma_1 \otimes p_0, \sigma_2 \otimes p_\infty \rangle\rangle_{0,2}^{QG^{\theta,\varepsilon}} &= \sum_i \langle\langle \sigma_1, \frac{\gamma^i}{z-\psi} \rangle\rangle_{0,2}^{\theta,\varepsilon} \langle\langle \frac{\gamma_i}{-z-\psi}, \sigma_2 \rangle\rangle_{0,2}^{\theta,\varepsilon} \\ &= \langle \sigma_1, \sigma_2 \rangle + O(1/z). \end{aligned}$$

On the other hand, $\langle\langle \sigma_1 \otimes p_0, \sigma_2 \otimes p_\infty \rangle\rangle_{0,2}^{QG^{\theta,\varepsilon}}$ is well-defined without any localization with respect to z. Hence we conclude the following (for details, see the proof of Proposition 5.3.1 of [7]).

Proposition 4.2. *The operator* $(\mathbb{S}^{\theta,\varepsilon})^\star_{\mathbf{t}}(-z)$ *defined by*

$$(\mathbb{S}^{\theta,\varepsilon})^\star_{\mathbf{t}}(-z)(\gamma) = \sum_i \gamma^i \langle\langle \gamma_i, \frac{\gamma}{-z-\psi} \rangle\rangle_{0,2}^{\theta,\varepsilon}$$

is the inverse of $S^{\theta,\varepsilon}_{\mathbf{t}}(z)$*, i.e.,*

$$(\mathbb{S}^{\theta,\varepsilon})^\star_{\mathbf{t}}(-z) \circ \mathbb{S}^{\theta,\varepsilon}_{\mathbf{t}}(z) = \mathrm{Id}.$$

4.2. **The P-series.** For $\mathbf{t} = \sum_i t_i \tilde{\gamma}_i$, let

$$\begin{aligned} P^{\theta,\varepsilon}(\mathbf{t}, z) &:= \sum_i \gamma^i \langle\langle \gamma_i \otimes p_\infty \rangle\rangle_{0,1}^{QG^{\theta,\varepsilon}} \\ &= (\mathbb{S}^{\theta,\varepsilon})^\star_{\mathbf{t}}(-z)(\mathbb{J}^{\theta,\varepsilon}(\mathbf{t}, z)). \end{aligned} \tag{4.2.1}$$

The latter equality follows from the $\mathbb{C}^*$-localization factorization. From this and Proposition 4.2 we obtain the following analog of the Birkhoff factorization Theorem 5.4.1 of [7].

Proposition 4.3.

$$\mathbb{J}^{\theta,\varepsilon}(\mathbf{t}, z) = \mathbb{S}^{\theta,\varepsilon}_{\mathbf{t}}(z)(P^{\theta,\varepsilon}(\mathbf{t}, z)).$$

Note that Proposition 4.3 together with (3.2.2) and (4.1.2) implies that

$$P^{\theta,\varepsilon}(\mathbf{t}, z) = \mathbb{1} + O(q).$$

4.3. **Polynomiality.** For $\mu \in (W/\!\!/\mathbf{G})^{\mathbf{T}}$, let

$$\delta_\mu := (\iota_\mu)_*[\mu] \in H^*_{\mathbf{T}}(W/\!\!/\mathbf{G}, \mathbb{Q})$$

where ι_μ is the $\mathbf{T}$-equivariant closed immersion $\{\mu\} \hookrightarrow W/\!\!/\mathbf{G}$. Let

$$\mathbb{S}^{\theta,\varepsilon}_\mu(q, \mathbf{t}, z) := \langle \mathbb{S}^{\theta,\varepsilon}_{\mathbf{t}}(z)(\gamma), \delta_\mu \rangle,$$

for

$$\gamma = \sum_\beta q^\beta \gamma_\beta, \quad \gamma_\beta \in H^*_{\mathbf{T},\mathrm{loc}}(W/\!\!/\mathbf{G})[z].$$

Lemma 4.4. *For each fixed point $\mu \in (W/\!\!/\mathbf{G})^{\mathbf{T}}$, the product series*

$$\mathbb{S}^{\theta,\varepsilon}_\mu(q, \mathbf{t}, z)\mathbb{S}^{\theta,\varepsilon}_\mu(qe^{-zyL_\theta}, \mathbf{t}, -z)$$

has no pole at $z = 0$.

Proof. The proof is identical to the proof of Lemma 7.6.1 of [7]. □

4.4. **Comparison of S-operators.** It is obvious from definitions that the stability condition $(\infty \cdot \theta_0, 1)$ gives the usual moduli spaces of stable maps to $W/\!\!/\mathbf{G}$ (or to $W/\!\!/\mathbf{G} \times \mathbb{P}^1$ for the graph spaces), hence the resulting theory is the Gromov-Witten theory of $W/\!\!/\mathbf{G}$. We will simply write $(\infty, 1)$ for this stability condition. This is justified, since the theory is independent on the choice of θ_0, as long as we stay in the same GIT chamber for the action of $\mathbf{G}$ on W.

Conjecture 4.5. *Let (θ, ε) be arbitrary, including all asymptotic cases. Then*

(1)

$$\mathbb{S}^{\theta,\varepsilon}_{\mathbf{t}}(\mathbb{1}) = \mathbb{S}^{(\infty,1)}_{\tau(\mathbf{t})}(\mathbb{1})$$

with

$$\tau(\mathbf{t}) := \kappa(\mathbf{t}) + \sum_{\beta \neq 0} q^\beta \sum_i \gamma_i \langle\!\langle \gamma^i, \mathbb{1} \rangle\!\rangle^{\theta,\varepsilon}_{0,2,\beta}.$$

(2) *For $\mathbf{t} := \sum_i t_i \tilde{\gamma}_i$, there are unique*

$$P^{(\infty,1),\theta,\varepsilon}(\mathbf{t}, z) = \mathbb{1} + O(q) \in H^*_{\mathbf{T},\mathrm{loc}}(W/\!\!/\mathbf{G})[z][[q, t_j]],$$
$$\tau^{(\infty,1),\theta,\varepsilon}(\mathbf{t}) = \kappa(\mathbf{t}) + O(q) \in H^*_{\mathbf{T},\mathrm{loc}}(W/\!\!/\mathbf{G})[[q, t_j]]$$

such that

$$\mathbb{S}^{\theta,\varepsilon}_{\mathbf{t}}(z)(P^{\theta,\varepsilon}(\mathbf{t}, z)) = \mathbb{S}^{(\infty,1)}_{\tau^{(\infty,1),\theta,\varepsilon}(\mathbf{t})}(z)(P^{(\infty,1),\theta,\varepsilon}(\tau^{(\infty,1),\theta,\varepsilon}(\mathbf{t}), z)). \tag{4.4.1}$$

Just as in [7, Lemma 6.4.1], one can recursively construct uniquely determined series $P^{(\infty,1),\theta,\varepsilon}(\mathbf{t},z)$ and $\tau^{(\infty,1),\theta,\varepsilon}(\mathbf{t})$ with the required q-asymptotics, and which satisfy equation (4.4.1) *modulo* $1/z^2$. The content of part (2) of Conjecture 4.5 is that equality modulo $1/z^2$ suffices to force the equality to all orders in $1/z$. Note that when combined with Proposition 4.3, part (2) implies Conjecture 3.2.

Theorem 4.6. *Suppose that the induced* $\mathbf{T}$*-action on* $W/\!\!/\mathbf{G}$ *has only isolated* $\mathbf{T}$*-fixed points. Then Conjecture 4.5 (1) holds true.*

Further, if in addition $W/\!\!/\mathbf{G}$ *has only isolated 1-dimensional* $\mathbf{T}$*-orbits, then Conjecture 4.5 (2) holds true.*

Proof. The proof of the first statement is identical with the proof of Theorem 7.3.1 of [7], while the proof of the second statement is identical with the proof of Theorem 7.3.4 of [7]. □

Now the proof of Theorem 3.3 follows from Proposition 4.3 and Theorem 4.6.

4.5. **Non-equivariant limit.** If $W/\!\!/\mathbf{G}$ is projective then one can work with the *non-localized* equivariant cohomology ring $H^*_{\mathbf{T}}(W/\!\!/\mathbf{G},\mathbb{Q})$, the Poincaré pairing with values in $\mathbb{Q}[\lambda_1,\dots,\lambda_r]$, and the Novikov ring $\Lambda_{\mathbf{T}}$. The objects $\mathbb{J}^{\theta,\varepsilon}$, $\mathbb{S}^{\theta,\varepsilon}_{\mathbf{t}}$, $P^{\theta,\varepsilon}$, $\tau(\mathbf{t})$, $P^{(\infty,1),\theta,\varepsilon}$, and $\tau^{(\infty,1),\theta,\varepsilon}$ reduce to their non-equivariant counterparts upon setting $\lambda_1=\cdots=\lambda_r=0$.

5. Explicit Formula for the fully asymptotic stability condition

5.1. **$\mathbb{I}$-function.** Other than the Gromov-Witten chamber $(\theta,\varepsilon)=(\infty,1)$, the most interesting case from a computational viewpoint is the opposite asymptotic case $(\theta,\varepsilon)=(0+,0+)$ (again, the theory is independent on the choice of character in a given GIT chamber, so we drop θ_0 from the notation). The main reason is that $QG^{0+,0+}_{0,0|k,\beta}([W/\mathbf{G}])$ is isomorphic to

$$QG^{0+,0+}_{0,0,\beta}([W/\mathbf{G}])\times(\mathbb{P}^1)^k.$$

The space $QG^{0+,0+}_{0,0,\beta}([W/\mathbf{G}])$ coincides with $\mathrm{Qmap}_{0,0}(W/\!\!/_\theta\mathbf{G},\beta;\mathbb{P}^1)$ defined in [10, §7.2] and was denoted by $QG_{0,0,\beta}(W/\!\!/_\theta\mathbf{G})$ in [7, §2.6]. Further, as we already noted earlier

$$F_{k,\beta}=F_\beta\times 0^k,$$

where $F_\beta=F_{0,\beta}$ is the distinguished $\mathbb{C}^*$-fixed locus in $QG^{0+,0+}_{0,0,\beta}([W/\mathbf{G}])$.

Denote

$$\mathbb{I}=\mathbb{I}_{W/\!\!/_\theta\mathbf{G}}(q,\mathbf{t},z):=\mathbb{J}^{0+,0+}(q,\mathbf{t},z). \tag{5.1.1}$$

In this paper we will call $\mathbb{I}_{W/\!\!/_\theta \mathbf{G}}(q,\mathbf{t},z)$ *the big* $\mathbb{I}$*-function of* $W/\!\!/_\theta\mathbf{G}$. This differs from the terminology in [10, 7]. The specialization

$$\mathbb{I}(q,0,z)=\sum_\beta q^\beta \mathbb{I}_\beta(z)$$

is called the small $\mathbb{I}$-function of $W/\!\!/_\theta\mathbf{G}$ (this terminology *does* agree with the one in [7, 8]). We have

$$\mathbb{I}_\beta(z)=(ev_\bullet)_*\mathrm{Res}_{F_\beta}[QG_{0,0,\beta}(W/\!\!/_\theta\mathbf{G})]^{\mathrm{vir}}. \tag{5.1.2}$$

As is well-known, these push-forwards of residues can often be explicitly calculated in closed form. For example, the case of toric varieties goes back to Givental, [15], see also [6, §7.2] for an exposition. Type A flag varieties, which are examples with non-abelian $\mathbf{G}$, are treated in [3, 4]. For more on the non-abelian case, see the forthcoming note [9].

The goal of this section is to find explicit formulas for the big $\mathbb{I}$-functions for some $(W,\mathbf{G},\theta)$. To emphasize the role of class β, we write

$$\hat{ev}_\beta=\hat{ev}_j : F_\beta\to[W/\mathbf{G}].$$

Note that these evaluation maps do not depend on the choice of j since all marked points are concentrated on $0\in\mathbb{P}^1$. It follows that

$$\mathbb{I}(q,\mathbf{t},z)=\sum_\beta q^\beta(ev_\bullet)_*(\exp(\hat{ev}_\beta^*(\mathbf{t})/z)\cap\mathrm{Res}_{F_\beta}[QG_{0,0,\beta}(W/\!\!/_\theta\mathbf{G})]^{\mathrm{vir}}).$$

Suppose that for some $\gamma_{i,\beta}(z)\in H^*(W/\!\!/_\theta\mathbf{G})\otimes\mathbb{Q}[z]$,

$$(ev_\bullet)^*\gamma_{i,\beta}(z)=(\hat{ev}_\beta^*(\tilde\gamma_i)).$$

Then by the projection formula, the big $\mathbb{I}$-function becomes

$$\sum_\beta e^{\sum_i t_i\gamma_{i,\beta}(z)/z}q^\beta\mathbb{I}_\beta(z)\quad\text{for}\quad \mathbf{t}=\sum t_i\tilde\gamma_i. \tag{5.1.3}$$

Whenever the small I-function is known, to obtain an explicit formula for $\mathbb{I}$ it remains to find explicitly such classes $\gamma_{i,\beta}(z)$.

Remark 5.1. By Theorem 3.3, the big $\mathbb{I}$-function (5.1.3) is on the Lagrangian cone of the Gromov-Witten theory of $W/\!\!/_\theta\mathbf{G}$ whenever the $\mathbf{T}$ action has isolated fixed points and isolated 1-dimensional orbits. This statement is presumably related to Woodward's result in [23, Theorem 1.6].

5.2. **Description of ev.** Let $A(W)$ be the affine coordinate ring of W and let ζ_0, ζ_1 be the homogeneous coordinates of $\mathbb{P}^1$ defining $0 \in \mathbb{P}^1$ by the equation $\zeta_0 = 0$.

For a sufficiently large and divisible integer m, the character $m\theta$ defines a morphism

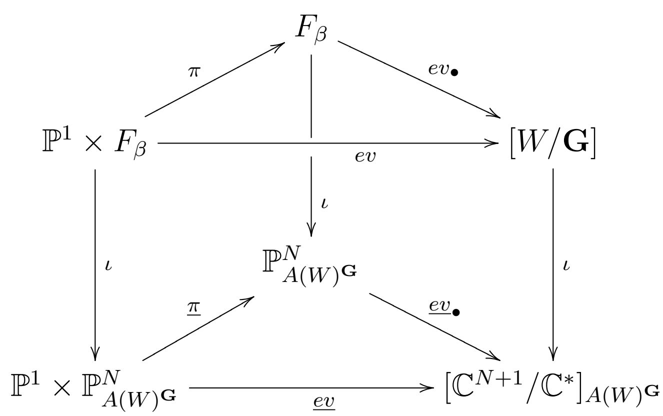
$$\iota : [W/\mathbf{G}] \to [\mathbb{C}^{N+1}/\mathbb{C}^*]_{A(W)^{\mathbf{G}}} := [\mathrm{Spec}(A(W)^{\mathbf{G}}) \times \mathbb{C}^{N+1}/\mathbb{C}^*]$$

whose restriction $W/\!\!/\mathbf{G} \to \mathbb{P}^N_{A(W)^{\mathbf{G}}}$ is an embedding (see [7, §3.1]). Let $d := \beta(L_{m\theta})$. Recall that

$$QG_{0,0,d}(\mathbb{C}^{N+1}/\!\!/_{\mathrm{id}}\mathbb{C}^*) = \mathbb{P}(\mathrm{Sym}^d((\mathbb{C}^2)^\vee) \otimes \mathbb{C}^{N+1}),$$

and that its $\mathbb{C}^*$-fixed distinguished part F_d is $\mathbb{P}((\mathbb{C} \cdot x^d) \otimes \mathbb{C}^{N+1}) = \mathbb{P}^N$ (see [14]).

Consider now the following natural diagram



where

- the vertical morphisms are induced from ι (abusing notation, we denote all of them also by ι);
- $ev, \underline{ev}$ are the universal evaluation maps;
- $ev_\bullet, \underline{ev}_\bullet$ are evaluation maps at the generic point of $\mathbb{P}^1$;
- $\pi, \underline{\pi}$ are projections.

All side square faces are commutative but the upper and the lower triangle faces need not be commutative.

Let $w_0, ..., w_N$ be the homogeneous coordinates of $\mathbb{P}^N$. On the stack quotient $[\mathbb{C}^{N+1}/\mathbb{C}^*]_{A(W)^{\mathbf{G}}}$ we have the invertible sheaf $\mathcal{O}_{[\mathbb{C}^{N+1}/\mathbb{C}^*]_{A(W)^{\mathbf{G}}}}(1)$ attached to the character id. Let $\mathbb{C}_{nz}$ denote the $\mathbb{C}^*$-representation space given by the character $nz = n\mathrm{id}$.

The map $\underline{ev}$ is defined by the line bundle $\mathcal{O}_{\mathbb{P}^1}(d) \boxtimes \mathcal{O}_{\mathbb{P}^N_{A(W)^{\mathbf{G}}}}(1)$ together with sections $\zeta_0^d \boxtimes w_i$, $i = 0, ..., N$. Therefore as $\mathbb{C}^*$-equivariant coherent sheaves

$$\begin{aligned} \underline{ev}^*(\mathcal{O}_{[\mathbb{C}^{N+1}/\mathbb{C}^*]_{A(W)^{\mathbf{G}}}}(1)) &= \mathcal{O}_{\mathbb{P}^1}(d)\boxtimes \mathcal{O}_{\mathbb{P}^N_{A(W)^{\mathbf{G}}}}(1) \\ &= \mathcal{O}_{\mathbb{P}^1}(d)\boxtimes \underline{ev}_\bullet^*\mathcal{O}_{\mathbb{P}^N_{A(W)^{\mathbf{G}}}}(1), \end{aligned} \tag{5.2.1}$$

where $\mathcal{O}_{\mathbb{P}^1}(d)|_0=\mathbb{C}_{dz}$ and $\underline{ev}_\bullet^*\mathcal{O}_{\mathbb{P}^N_{A(W)^{\mathbf{G}}}}(1)$ has the trivial $\mathbb{C}^*$-equivariant structure.

Lemma 5.2. *The following equality holds in* $\mathrm{Pic}_{\mathbb{C}^*}(F_\beta)_{\mathbb{Q}}$*:*

$$\hat{ev}_\beta^*(L_\theta)=ev_\bullet^*(L_\theta)\boxtimes\mathbb{C}_{\beta(L)z},$$

where the $\mathbb{C}^*$*-action on* $ev_\bullet^*(L_\theta)$ *is trivial.*

Proof. We take ι^* on (5.2.1) and use $\iota^*\mathcal{O}_{\mathbb{P}^N_{A(W)^{\mathbf{G}}}}(1)=L_\theta^{\otimes m}$ to conclude the proof. □

Remark 5.3. Let θ' be another character in the same GIT chamber as θ. Since the moduli spaces of weighted stable quasimaps for the $((0+)\cdot\theta,0+)$ and $((0+)\cdot\theta',0+)$ stability conditions also coincide, we conclude that Lemma 5.2 also applies to $L_{\theta'}$. If the GIT chamber has dimension equal to the rank of the group of rational characters $\chi(\mathbf{G})\otimes\mathbb{Q}$, then it contains a basis of $\chi(\mathbf{G})\otimes\mathbb{Q}$ and therefore Lemma 5.2 holds for any character of $\mathbf{G}$ up to torsion.

5.3. **Examples.** If W is a vector space and $\mathbf{G}\cong(\mathbb{C}^*)^s$ is a torus, so that $W/\!\!/_\theta\mathbf{G}$ is a nonsingular toric variety, then $H^*([W/\mathbf{G}])$ is a polynomial algebra over $\mathbb{Q}$, with generators $c_1(L_{\eta_i})$ corresponding to a $\mathbb{Q}$-basis $\{\eta_1,\dots,\eta_s\}$ of $\chi(\mathbf{G})\otimes\mathbb{Q}$. By Remark 5.3, for any polynomial $p(c_1(L_{\eta_1}),\dots,c_1(L_{\eta_s}))$ we have

$$\hat{ev}_\beta^*p(c_1(L_{\eta_1}),\dots,c_1(L_{\eta_s}))=ev_\bullet^*p(c_1(L_{\eta_1})+\beta(L_{\eta_1})z,\dots,c_1(L_{\eta_s})+\beta(L_{\eta_s})z).$$

In particular, the classes $\gamma_{i,\beta}(z)$, and therefore the big $\mathbb{I}$-functions, are explicitly known for toric varieties. By considering twisted theories, the same is true for complete intersections in toric varieties as well. We exemplify with the case of projective spaces.

Let H denote the hyperplane class of $\mathbb{P}^n=\mathbb{C}^{n+1}/\!\!/\mathbb{C}^*$. In this case, applying Lemma 5.2 to (5.1.3) and using the formula for its small $\mathbb{I}$-function from [14], we obtain

$$\mathbb{I}_{\mathbb{C}^{n+1}/\!\!/\mathbb{C}^*}(q,\mathbf{t},z)=\sum_{d=0}^{\infty}q^d\frac{\exp(\sum_{i=0}^n t_i(H+dz)^i/z)}{\prod_{k=1}^d(H+kz)^{n+1}}.$$

By the non-equivariant specialization of Theorem 3.3, $\mathbb{I}_{\mathbb{C}^{n+1}/\!\!/\mathbb{C}^*}(q,\mathbf{t},z)$ is on the Lagrangian cone of the Gromov-Witten theory of $\mathbb{P}^n$.

More generally, let $E = \mathbb{C}$ with weight a positive integer l be the twisting factor, so that $\underline{E}|_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(l)$. With this setting,

$$\mathbb{I}^E_{\mathbb{C}^{n+1}/\!\!/\mathbb{C}^*}(\mathbf{t}) = \sum_{d=0}^{\infty} q^d \frac{\exp(\sum_{i=0}^n t_i(H+dz)^i/z)}{\prod_{k=1}^d (H+kz)^{n+1}} \prod_{k=0}^{ld}(lH+kz). \tag{5.3.1}$$

By Theorem 3.3, the Gromov-Witten E-twisted J-function of $\mathbb{P}^n$ is related with $\mathbb{I}^E_{\mathbb{C}^{n+1}/\!\!/\mathbb{C}^*}$ via a Birkhoff factorization (see [13] for the Birkhoff factorization procedure). Recall that the E-twisted J-function is essentially the usual J-function of a hypersurface of degree l in $\mathbb{P}^n$.

Note that RHS of (5.3.1) can be expressed as

$$\left(\exp(\sum_{i=0}^n \frac{t_i}{z}(zq\frac{\partial}{\partial q} + H)^i)\right) \mathbb{I}^E_{\mathbb{C}^{n+1}/\!\!/\mathbb{C}^*}(0).$$

This latter expression is already considered as a special case by Iritani in [18, Example 4.14] for a reconstruction of quantum D-modules.

Remark 5.4. Recent work by Coates, Corti, Iritani, and Tseng in [11, 12] introduces the so-called S-extended I-function of a toric DM stack $\mathcal{X} = [(\mathbb{C}^N)^{ss}(\theta)/(\mathbb{C}^*)^r]$ and proves that it lies on the Lagrangian cone of the Gromov-Witten theory of $\mathcal{X}$. In examples, see [12], by choosing the extending set S carefully, one can extract sufficient information from the S-extended I-function to recover the big J-function of $\mathcal{X}$.

From the perspective of our paper (generalized to orbifold GIT targets in [5]), the S-extension amounts to changing the GIT presentation of the toric target $\mathcal{X}$ as $[(\mathbb{C}^{N+|S|})^{ss}(\theta')/(\mathbb{C}^*)^{r+|S|}]$, and the S-extended I-function of [11, 12] coincides with the big $\mathbb{I}$-function of ours (corresponding to the *new* GIT presentation) *restricted* to $\mathbf{t} = \sum t_i\tilde{\gamma}_i$ with $\gamma_i \in H^{\leq 2}(\mathcal{X})$. The additional parameters of the S-extended I-function of [11] are identified with the additional "ghost" Novikov variables (see [8, §5.9.2]) of the quasimap theory for the new GIT presentation.

Put it differently, the S-extended I-function of [11] is exactly *Givental's small I-function* for the quasimap theory of $(\mathbb{C}^{N+|S|}, (\mathbb{C}^*)^{r+|S|}, \theta')$, as defined e.g., in equation (7.3.2) of [10] for the manifold case.

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