FUNDAMENTAL FACTORIZATION OF A GLSM
PART I: CONSTRUCTION

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Abstract. We define enumerative invariants associated to a hybrid Gauged Linear Sigma Model. We prove that in the relevant special cases these invariants recover both the Gromov–Witten type invariants defined by Chang–Li and Fan–Jarvis–Ruan using cosection localization as well as the FJRW type invariants constructed by Polishchuk–Vaintrob. The invariants are defined by constructing a "fundamental factorization" supported on the moduli space of Landau–Ginzburg maps to a convex hybrid model. This gives the kernel of a Fourier–Mukai transform; the associated map on Hochschild homology defines our theory.

Contents

0. Introduction

0.1. **Background.** In the last quarter century, enumerative geometry has witnessed a dramatic transformation, due to a tremendous influx of ideas coming from string theory. The development of Gromov–Witten theory for smooth algebraic varieties is one such example. These results have revealed two important facts about the field, which were not previously apparent but are now crucial to its study. First, that certain enumerative invariants of a smooth variety have a rich recursive structure when organized correctly. For curve counting theories such as Gromov–Witten theory, the invariants form what is known as a Cohomological Field Theory (CohFT). Second, the enumerative invariants for *different spaces* are often related to each other in dramatic and surprising ways. These correspondences take many forms, with most originating from physical considerations. One of the first and most striking instances of both of the phenomena described above was the celebrated mirror theorem, first conjectured by Candelas et. al. [CdlOGP91] and subsequently proven by Givental [Giv96]. In this theorem, the rational curve counts of a quintic hypersurface Q in $\mathbb{P}^4$ were shown to be dictated by the variation of Hodge structure of a different space, the so-called "mirror manifold" to the quintic.

In attempting to realize the full scope of mirror symmetry and other related correspondences, one is led to consider more exotic geometries generalizing the notion of a smooth manifold or variety. For instance the "mirror" to a smooth variety may not itself be a smooth variety but rather an *orbifold*, i.e. a smooth Deligne–Mumford stack. The Gromov–Witten theory of orbifolds has been an active area of study in the last decade [CR04, CR02, AGV08]. In another direction, again inspired by mirror symmetry considerations [Wit97], one can generalize the idea of a smooth variety by adding the data of a function, or *potential*. A space X together with a potential $w : X \to \mathbb{A}^1$ is known as a Landau–Ginzburg model (LG model).

For example, the space $T = \mathrm{tot}(\mathcal{O}_{\mathbb{P}^4}(-5))$ together with the potential

$$w = p \sum_{i=1}^{5} x_i^5$$

is the LG model realization of the quintic hypersurface $Q = \{\sum_{i=1}^5 x_i^5 = 0\} \subset \mathbb{P}^4$ (See Example 1.4.7 for details). Note that the degeneracy locus of w in T is exactly the hypersurface Q. A suitably defined curve-counting theory for the LG model (T, w) should agree with the Gromov–Witten theory for Q (see [CL11] for such a result). In certain special cases, LG models have also been studied by mathematicians in more familiar contexts. For instance given a polynomial $p : \mathbb{A}^N \to \mathbb{A}^1$ with an isolated singularity at the origin, many of the invariants of the singularity defined by p can be equally considered as classical invariants of the LG model $(\mathbb{A}^N, p)$. Thanks to pioneering work of Witten [Wit93] and a rigorous mathematical construction of Fan, Jarvis, and Ruan [FJR13, FJR07] and Polishchuk and Vaintrob [PV16], an enumerative theory known as FJRW theory has been defined for these *affine* LG models, in analogy with Gromov–Witten theory.

A particularly useful class of Landau–Ginzburg models are known as Gauged Linear Sigma Models (GLSMs). A GLSM is defined in part by a vector space V together with the action of a group G on V, and a G-invariant function $w : V \to \mathbb{A}^1$ (see Definition 1.1.1 for details). The space X is then given as a GIT quotient of V by G, and the function w descends to a potential $\bar{w} : X \to \mathbb{A}^1$. In fact both the quintic hypersurface Q "=" (T, w) and the affine model $(\mathbb{A}^N, p)$ are particular examples of GLSMs. Hence both Gromov–Witten theory (of hypersurfaces) and FJRW theory should be special cases of a more general but currently elusive curve counting theory for GLSMs. For a given GLSM, this theory should have the structure of a cohomological field theory.

In a major recent advance, Fan, Jarvis, and Ruan [FJR17] have begun to construct such a theory. More precisely, they construct a collection of moduli spaces associated to a GLSM, the so-called moduli of *Landau–Ginzburg stable maps* (Definition 1.3.1), and prove these spaces are Deligne–Mumford stacks. In addition, in the case of "compact type" insertions (see [FJR17, Definition 4.1.4]), they are able to endow these spaces with a virtual cycle defined on a proper substack. Enumerative invariants can then be defined as integrals over this cycle. This breakthrough is the first appearance of a theory of general GLSM invariants in mathematics. Unfortunately, certain technical constraints place restrictions on when these methods can be applied, and the technique yields only a partially defined theory.

0.2. **Construction.** The primary goal of the current project is to construct an enumerative curve counting theory for GLSMs which both is purely algebraic and defines a full cohomological field theory. In the current paper, we restrict our attention to a class of GLSMs called *convex hybrid models*. The theory is constructed in terms of the derived category of factorizations. This work forms the bulk of the paper, and is completed in §5 (see Definition 5.5.1).

Our methods are inspired by Polishchuk and Vaintrob's innovative use of matrix factorizations to give an algebraic construction of the FJRW theory of affine LG models [PV16]. The current work extends their methods to the general setting of hybrid model GLSMs. As with Polishchuk–Vaintrob, the construction takes place at the level of derived categories; for LG models these are categories of factorizations (also known as matrix factorizations).

Given a hybrid model GLSM, we construct a family of smooth moduli spaces which contain the moduli of LG stable maps as a closed substack. On these moduli spaces we define a *fundamental factorization*, an object in the derived category of factorizations with support contained in the space of LG maps described above.

As our virtual cycle is lifted to a categorical object, so too are our generalized enumerative invariants. Namely, the fundamental factorization serves as the kernel of a Fourier–Mukai transform - a functor from the derived category of factorizations of a hybrid model to the derived category of the moduli space of curves, $\overline{\mathcal{M}}_{g,r}$. This functor can be viewed as a categorical lift of the usual description of Gromov–Witten-type invariants in terms of cohomological integral transforms. Indeed, the passage to Hochschild homology (see Section 2.4) recovers the more familiar description, after application of a suitable Hochschild–Kostant–Rosenberg isomorphism identifying Hochschild homology with cohomology.

0.3. **Comparisons.** As mentioned above, both the Gromov–Witten theory of a complete intersection (in a convex GIT quotient of the form $[V/\!\!/_\theta G]$), as well as the FJRW theory of a singularity $w : [\mathbb{A}^N/G] \to \mathbb{A}^1$ are special cases of the hybrid model GLSMs we consider. Indeed the term *hybrid model* refers to a "hybrid" between Gromov–Witten and FJRW theory. As such, a key test of the validity of our construction is to show that in these two opposite limiting cases our invariants agree with the previously defined Gromov–Witten or FJRW invariants. These comparison results are proven in §6, Theorems 6.1.8 and 6.2.3 respectively.

There are many examples of different GLSMs which define the same LG model. More precisely, one may easily construct a pair of GLSMs involving different vector spaces, groups and functions, but such that the corresponding GIT quotients together with their induced potentials are isomorphic. In this case it is natural to ask if the corresponding GLSM invariants can be identified with one another. In the final section we consider so-called affine phase GLSMs, i.e. abelian GLSMs such that the corresponding GIT quotient is isomorphic to a singularity $w : [\mathbb{A}^N/G] \to \mathbb{A}^1$. We prove (Theorem 6.3.3) that the invariants of any such affine phase agree with the FJRW invariants of the singularity as defined in [PV16].

0.4. **Future work.** There are two technical steps in the construction presented herein which impose restrictions on the types of GLSMs that we currently treat. They are:

- to embed Fan–Jarvis–Ruan's moduli space of LG maps (Definition 1.3.1) into a smooth Deligne–Mumford stack, denoted $\square$, which has *quasi-projective coarse moduli*;
- to realize this embedding as the zero locus of a section of a certain vector bundle on $\square$.

The first step forces us to restrict our attention to the class of so-called hybrid model GLSMs, due to the projectivity requirement. To obtain the second step, we impose the convexity condition of Definition 4.1.1. These restrictions are in fact not strictly necessary, but we leave the more general case to a later paper for two reasons. First, we wish to streamline the exposition here as much as possible. Second, the construction described here is important in its own right, as it is necessary for our comparison with Gromov–Witten theory in § 6.

Both Gromov–Witten theory and FJRW theory are known to have the structure of a cohomological field theory. In fact the general GLSM invariants constructed here also form a cohomological field theory. For "brevity" we have chosen to save this result for the sequel [CFG$^+$].

Finally, we hope that our construction will shed new light on the connections between previously defined enumerative theories. For example, our level of generality should be readily applicable to wall crossing results such as the LG/CY correspondence and should help to service the abundance of recent developments in this area, see e.g. [CR10, AS15, LPS16, GR17, CJR17].

0.5. **Structure of the paper.** §1 is in many ways an expanded introduction. We define GLSMs and hybrid models, give a general overview of our construction of an enumerative theory for GLSMs, and review the description of the moduli space constructed in [FJR17].

In §2, we give an overview of categories of factorizations. It summarizes the constructions of the (derived) pushforward, pullback, tensor product (see §2.2), and Fourier–Mukai transforms, as well as Hochschild homology (see §2.4). Moreover, we set the stage for our comparison with Gromov–Witten theory, with a collection of results relating various cohomological and categorical integral transforms.

In §3, we construct a factorization associated to families of curves equipped with GLSM data. We call this factorization the Polishchuk–Vaintrob (PV) factorization and determine its properties. In order to define it, we require three conditions (see §3.1 and §3.2). We prove that Condition 1 is automatic (see §3.4) and that Conditions 2 and 3 are satisfied assuming that the base is a smooth Deligne-Mumford stack which is projective over an affine; this is called Assumption ($\star$) (see §3.5). We establish that the PV factorization is supported along the critical locus of the superpotential in §3.6. Then finally in §3.7, we discuss a $\mathbb{C}^*$-equivariant version of the PV factorization necessary for our construction.

§4 is dedicated to embedding the space of LG maps in a smooth Deligne–Mumford stack, and proving that the necessary projectivity assumptions hold on this ambient space.

Finally in §5, we define our GLSM invariants. We gather results from previous sections to construct a "fundamental factorization" supported on the moduli space of LG maps to the critical locus. This forms the kernel of an integral transform, which is used to define our invariants. In this section also we prove that these invariants are independent of the choices made in the construction.

In §6, we compare the GLSM invariants of §5 with other previously defined theories, such as a cosection localized version of Gromov–Witten theory of complete intersection varieties and the quantum singularity theory (or FJRW theory).

0.6. **Conventions and notations.** We work in the algebraic category and over the field $\mathbb{C}$ of complex numbers. The algebraic stacks considered in this paper are noetherian and semi-separated over the ground field unless otherwise stated.

0.7. *Acknowledgements.* I. Ciocan-Fontanine was partially supported by NSF grant DMS-1601771. D. Favero was partially funded by the Natural Sciences and Engineering Research Council of Canada and Canada Research Chair Program under NSERC RGPIN 04596 and CRC TIER2 229953. J. Guéré was first funded by the Einstein Foundation at the Humboldt University of Berlin and then by the University of Grenoble–Alpes. B. Kim would like to thank K. Chung, T. Jarvis, T. Kim, S. Lee, A. Kresch, M. Miura, and A. Polishchuk for helpful discussions. M. Shoemaker would like to thank Y. Ruan and T. Jarvis for teaching him the GLSM framework and I. Shipman for many useful conversations on factorizations. M. Shoemaker was partially supported by NSF grant DMS-1708104. I. Ciocan-Fontanine and D. Favero would also like to thank the Korea Institute for Advanced Study for their superb and abundant hospitality throughout the process of developing this work. All authors are also grateful to the University of Minnesota's Department of Mathematics for the use of their facilities during several extended collaborative meetings.

1. Overview of the construction

1.1. **Input data.** A Gauged Linear Sigma Model, or GLSM, is a collection of data describing a GIT quotient $[V/\!\!/_\theta G]$ of a vector space V together with a superpotential $w\colon [V/\!\!/_\theta G] \to \mathbb{A}^1$ and an "R-charge". The "open string B-model" theory of a GLSM is given by the derived categories of factorizations. These categories have received a lot of attention in recent years (e.g. [BFK14] [PV16]). We discuss these categories in detail in §2 below. A "closed string A-model" theory with target a GLSM was introduced by Fan, Jarvis, and Ruan [FJR17]. We recall briefly the main points.

Definition 1.1.1. *The GLSM input data $(V, G, \mathbb{C}^*_R, \theta, w)$ consists of:*

(a) *a $\mathbb{Z}$-graded vector space, $V = \oplus_{n\in\mathbb{Z}} V_n$, with grading induced by the action (called the R-charge) of a one-dimensional torus $\mathbb{C}^*_R \subseteq GL(V)$;*
(b) *an action by a linearly reductive group $G \subseteq GL(V)$;*
(c) *a choice of character $\theta \in \widehat{G}_{\mathbb{Q}} := \mathrm{Hom}(G, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}$;*
(d) *a G-invariant polynomial function $w\colon V \to \mathbb{A}^1$, homogeneous of degree $\mathbf{deg} > 0$ with respect to the grading action $\mathbb{C}^*_R$.*

The actions of G and $\mathbb{C}^*_R$ are required to be compatible, that is, to satisfy

- *G and $\mathbb{C}^*_R$ commute: $g\lambda = \lambda g$ for all $g \in G$ and $\lambda \in \mathbb{C}^*_R$;*
- *$\langle J\rangle := G \cap \mathbb{C}^*_R \simeq \mu_{\mathbf{deg}}$.*

Here
$$J := (..., e^{2\pi\sqrt{-1}c_i/\mathbf{deg}}, ...)$$
is the diagonal element in $GL(V)$ given by the weights c_i of the $\mathbb{C}^*_R$-action on V. Moreover, we assume that there are no strictly semistable points for the linearization of the G-action on V given by θ:
$$V^{ss}(\theta) = V^s(\theta). \tag{1.1}$$

It follows from equation (1.1) that the GIT stack quotient

$$[V/\!\!/_\theta G] := [V^{ss}(\theta)/G]$$

is a smooth separated Deligne–Mumford stack, and that its coarse moduli space, $V/\!\!/_\theta G$, is projective over the affine scheme $\operatorname{Spec}(\operatorname{Sym}^\bullet(V^\vee))^G$.

The regular function $w\colon V \to \mathbb{A}^1$ is called the superpotential. The superpotential descends to a function on the GIT stack quotient, which, by abuse, we also denote $w\colon [V/\!\!/_\theta G] \to \mathbb{A}^1$. Define the closed substack

$$Z(dw) \subseteq [V^{ss}(\theta)/G],$$

as the critical locus of w, i.e., the zero locus of the section dw. We say the superpotential w is nondegenerate if $Z(dw)$ is proper over $\operatorname{Spec}(\mathbb{C})$.

Furthermore, we introduce the group

$$\Gamma := G \cdot \mathbb{C}^*_R \subseteq \operatorname{GL}(V).$$

By compatibility, the R-charge induces a well defined character

$$\chi\colon \quad \begin{aligned} \Gamma &\to \mathbb{C}^* \\ g\cdot\lambda &\mapsto \lambda^{\mathbf{deg}} \end{aligned}$$

for $g \in G$ and $\lambda \in \mathbb{C}^*_R$. We obtain the short exact sequence

$$1 \to G \to \Gamma \xrightarrow{\chi} \mathbb{C}^* \to 1. \tag{1.2}$$

Remark 1.1.2. In [FJR17], the definition of the A-theory of a GLSM also requires a choice of a good lift of θ to $\widehat{\Gamma}$, that is, the choice of a character $\nu \in \widehat{\Gamma}$ whose restriction to G is equal to θ and such that the Γ-semistable points for ν coincide with the G-semistable points for θ:

$$V^{ss}(\nu) = V^{ss}(\theta).$$

The existence of such a good lift is not automatic, but holds for almost all interesting examples; in particular, it will be automatic for the class of hybrid models that are the focus of this paper.

1.2. Landau–Ginzburg quasimaps.

Let $(V, G, \mathbb{C}^*_R, \theta, \nu, w)$ be GLSM input data as above. Fix $g, r \geq 0$.

Definition 1.2.1. *A prestable genus-g, r-pointed Landau–Ginzburg (LG) quasimap to $[V/\!\!/_\theta G]$ over a scheme S consists of:*

(a) *a prestable genus-g, r-pointed orbicurve $(\pi : \mathfrak{C} \to S, \mathscr{G}_1, \ldots, \mathscr{G}_r)$ with a section $S \to \mathscr{G}_i$ of the gerbe marking $\mathscr{G}_i$ for $1 \leq i \leq r$;*
(b) *a principal Γ-bundle $\mathcal{P} : \mathfrak{C} \to B\Gamma$ over $\mathfrak{C}$;*
(c) *a section $\sigma : \mathfrak{C} \to \operatorname{tot}\mathcal{V} := \mathcal{P} \times_\Gamma V$; and*
(d) *an isomorphism $\varkappa : \chi_*(\mathcal{P}) \to \mathring{\omega}^{\log}_{\mathfrak{C}}$,*

such that

- *the morphism of stacks $\mathcal{P} : \mathfrak{C} \to B\Gamma$ is representable;*
- *for each geometric fiber $\mathfrak{C}_s$ of $\mathfrak{C}$, the subset $B := \sigma^{-1}(\mathcal{V} \setminus (\mathcal{P} \times_\Gamma V^{ss}(\theta)))$ of $\mathfrak{C}_s$ is a finite set disjoint from the nodes and markings of $\mathfrak{C}_s$; the points of B are called the base-points of the LG quasimap.*

In item (4) of the above definition, $\omega_{\mathfrak{C}}^{\log}$ is the logarithmic relative canonical line bundle and $\mathring{\omega}_{\mathfrak{C}}^{\log}$ denotes the associated principal $\mathbb{C}^*$-bundle. The notation $\operatorname{tot}\mathcal{V}$ is explained as follows: the mixed construction $\mathcal{P}\times_\Gamma V = [(\mathcal{P}\times V)/\Gamma]$ defines a geometric vector bundle on $\mathfrak{C}$ and $\mathcal{V}$ denotes its locally free sheaf of sections.

There is a natural notion of isomorphism of LG quasimaps, analogous to that of quasimaps [CKM14] which we present now. Given a Landau–Ginzburg quasimap as above and a character $\eta\in\widehat{\Gamma}$, let $\mathcal{L}_\eta$ denote the line bundle

$$\mathcal{L}_\eta := \mathcal{P}\times_\Gamma \mathbb{C}(\eta). \tag{1.3}$$

If S is connected, the degree of the principal Γ-bundle $\mathcal{P}$ is an element $d_0\in\operatorname{Hom}_{\mathbb{Z}}(\widehat{\Gamma},\mathbb{Q})$ defined by

$$d_0(\eta)=\deg(\mathcal{L}_\eta)\in\mathbb{Q},$$

where $\deg(\mathcal{L}_\eta)$ is the degree of $\mathcal{L}_\eta$ on the fibers of $\mathfrak{C}$.

Furthermore, by Equation (1.2) we have the sequence:

$$0\to\operatorname{Hom}_{\mathbb{Z}}(\widehat{G},\mathbb{Q})\to\operatorname{Hom}_{\mathbb{Z}}(\widehat{\Gamma},\mathbb{Q})\xrightarrow{\chi_*}\operatorname{Hom}_{\mathbb{Z}}(\widehat{\mathbb{C}^*},\mathbb{Q})\to 0.$$

Since χ_* is an isomorphism when restricted to $\operatorname{Hom}_{\mathbb{Z}}(\widehat{\mathbb{C}}^*_R,\mathbb{Q})$, this sequence has a distinguished splitting. Let

$$q\colon \operatorname{Hom}_{\mathbb{Z}}(\widehat{\Gamma},\mathbb{Q})\to\operatorname{Hom}_{\mathbb{Z}}(\widehat{G},\mathbb{Q})$$

denote the map induced by this splitting, so that we have an isomorphism

$$(q,\chi_*):\operatorname{Hom}_{\mathbb{Z}}(\widehat{\Gamma},\mathbb{Q})\xrightarrow{\sim}\operatorname{Hom}_{\mathbb{Z}}(\widehat{G},\mathbb{Q})\times\operatorname{Hom}_{\mathbb{Z}}(\widehat{\mathbb{C}^*},\mathbb{Q}).$$

Definition 1.2.2. *The degree of an LG quasimap over a connected base is defined to be*

$$q(d_0)\in\operatorname{Hom}_{\mathbb{Z}}(\widehat{G},\mathbb{Q}).$$

We say an LG-quasimap over a scheme S has degree d if $q(d_0)=d$ for every connected component of S.

Remark 1.2.3. The reasoning for the definition above is as follows. Given an LG quasimap of degree $d_0\in\operatorname{Hom}_{\mathbb{Z}}(\widehat{\Gamma},\mathbb{Q})$, the image of d_0 under χ_* is determined by condition (d) of Definition 1.2.1. Thus for an LG quasimap, the degree d_0 as an element of $\operatorname{Hom}_{\mathbb{Z}}(\widehat{\Gamma},\mathbb{Q})$ can be recovered from its image $q(d_0)\in\operatorname{Hom}_{\mathbb{Z}}(\widehat{G},\mathbb{Q})$.

Let Z be a closed subscheme of V, invariant under the action of G. Let $\mathcal{Z}_\theta$ denote $[Z\cap V^{ss}(\theta)/G]$.

Definition 1.2.4. *A prestable LG quasimap to $\mathcal{Z}_\theta$ over S is a prestable LG quasimap to $[V/\!\!/_\theta G]$ satisfying the additional condition that the image of the associated Γ-equivariant map $[\sigma]:\mathcal{P}\to V$ lies in Z.*

1.3. **Stability conditions and moduli of stable LG maps.** In analogy with the theory of quasimaps to GIT quotients [CKM14], a family of stability conditions on LG quasimaps, indexed by a parameter $\varepsilon\in\mathbb{Q}_{>0}\cup\{0^+,\infty\}$ is introduced in [FJR17]. In this paper, we work with the asymptotic stability $\varepsilon=\infty$, which is directly analogous to the more familiar notion of stable maps.

Definition 1.3.1. *A prestable r-pointed Landau–Ginzburg quasimap to $[V/\!\!/_\theta G]$ over S is a stable LG map to $[V/\!\!/_\theta G]$ if for each geometric fiber C_s of $\mathfrak{C}$ the following two conditions hold:*

(a) *the set B of base-points is empty;*

(b) *the line bundle* $\omega_{\mathfrak{C}}^{\log} \otimes \mathcal{L}_\nu^{\otimes M}$ *is ample for* M *sufficiently large.*

The same two conditions define stable LG maps to $\mathcal{Z} = [Z \cap V^{ss}/G]$ *for any closed* G*-invariant subscheme* $Z \subseteq V$.

Note that stability depends on the good lift ν of θ.

Definition 1.3.2. *Given* g *and* r*, GLSM input data* $(V, G, \mathbb{C}^*_R, \theta, \nu, w)$*, a choice of* $Z \subseteq V$ *as above, and* $d \in \mathrm{Hom}_{\mathbb{Z}}(\widehat{G}, \mathbb{Q})$*, the moduli stack*

$$LG_{g,r}(\mathcal{Z}, d),$$

of genus-g*,* r*-pointed, degree* d *Landau–Ginzburg maps to* $\mathcal{Z}$ *is the stack parametrizing the degree* d *stable families of Definition 1.3.1.*

Theorem 1.3.3 (Fan–Jarvis–Ruan, Theorems 5.3.1 and 5.4.1)**.** *The moduli space* $LG_{g,r}(\mathcal{Z}, d)$ *is a separated Deligne–Mumford stack of finite type. When* $\mathcal{Z}$ *is proper over* $\mathrm{Spec}(\mathbb{C})$*,* $LG_{g,r}(\mathcal{Z}, d)$ *is proper as well.*

For convex hybrid models (see §1.4 and §4.1), we will prove further that $LG_{g,r}(\mathcal{Z}, d)$ is a global quotient stack with projective coarse moduli (Proposition 4.3.2).

1.4. **Hybrid models.** In this paper we focus on hybrid models. (In fact an additional technical assumption, called convexity over BG, will be required, see §4.1.) This class of GLSMs includes "geometric phases", such as complete intersections in convex varieties, as well as the "affine phases" of [FJR17, PV16].

Definition 1.4.1. *The* hybrid model input data $(V, G, \mathbb{C}^*_R, \theta, w)$ *is a GLSM input data as in Definiton 1.1.1, satisfying the following additional requirements:*

(a) *The graded vector space* V *decomposes as* $V = V_1 \oplus V_2 \cong \mathbb{C}^n \oplus \mathbb{C}^m$*, such that the* $\mathbb{C}^*_R$*-action is trivial on* V_1 *and has positive weights* $c_1, \ldots, c_m > 0$ *on* V_2*;*

(b) *The character* $\theta \in \widehat{G}_{\mathbb{Q}}$ *is such that* $V_1^s(\theta) = V_1^{ss}(\theta)$ *and* $V^{ss}(\theta) = V_1^{ss}(\theta) \times V_2$*;*

Notation 1.4.2. Let us denote

$$\begin{aligned} \mathcal{X} &:= [V_1 /\!\!/_\theta G] := [V_1^{ss}(\theta)/G], \\ \mathcal{T} &:= [V /\!\!/_\theta G] := [V^{ss}(\theta)/G], \\ \mathcal{Z} &:= Z(dw) \subseteq \mathcal{T}. \end{aligned}$$

Definition 1.4.3. *A* hybrid model *consists of input data* $(V = V_1 \oplus V_2, G, \mathbb{C}^*_R, \theta, w)$ *from above such that*

(a) $(\mathrm{Sym} V_1^\vee)^G = \mathbb{C}$ *(so that the stack* $\mathcal{X}$ *is projective);*

(b) G *and* $\mathbb{C}^*_R$ *are compatible; and*

(c) w *is nondegenerate and vanishes at* 0.

Note that, by requirement (a) of Definition 1.4.3, $\mathcal{X}$ is a smooth Deligne–Mumford stack whose coarse moduli is projective Furthermore, by requirement (b) of Definition 1.4.1, $\mathcal{T}$ is identified with the total space of a vector bundle (with fiber V_2) on the stack $\mathcal{X}$. Also, the superpotential w is nondegenerate if and only if we have $\mathcal{Z} \subseteq \mathcal{X} \subseteq \mathcal{T}$, where $\mathcal{X} \subseteq \mathcal{T}$ is the inclusion as the zero section.

Remark 1.4.4. In the hybrid model case, the choice of a good lift ν of θ always exists and is unique up to a multiple. Namely, after first replacing θ by a multiple, we can assume that θ is trivial on $\langle J\rangle = \langle(1,\dots,1,e^{2\pi i c_1/\deg},\dots,e^{2\pi i c_m/\deg})\rangle$. Hence, we can extend the character to Γ by sending $\mathbb{C}^*_R \subseteq \Gamma$ to 1. Note that the semi-stable locus of V remains unchanged in this process, meaning, in the language of [FJR17], that we indeed obtain a good lift. This is the reason for omitting the good lift from the notation for hybrid models.

Definition 1.4.5. (a) *A hybrid model GLSM $(V, G, \mathbb{C}^*_R, \theta, w)$ is called an affine phase if $[V_1 /\!\!/_\theta G] \cong BG'$ for some finite group G'.*

(b) *A hybrid model GLSM $(V, G, \mathbb{C}^*_R, \theta, w)$ is called a geometric phase if the group $\mathbb{C}^*_R$ acts on V_2 via the standard multiplication (so that $c_1 = c_2 = \dots = c_m = 1$), and w is a polynomial function which is linear on V_2, i.e.,*
$$w \in (V_2^\vee \otimes_{\mathbb{C}} \mathrm{Sym}^{\geq 1}(V_1^\vee))^G.$$
This implies $\mathbf{deg} = 1$, therefore $\langle J\rangle = 1$. Consider the vector bundle $\mathcal{E}$ (with fiber V_2) on $\mathcal{X}$, whose total space is $\mathcal{T}$. Since w is linear on V_2, it gives rise to a section $f \in H^0(\mathcal{X}, \mathcal{E}^\vee)$ of the dual vector bundle $\mathcal{E}^\vee$. Nondegeneracy of the hybrid model implies that f is a regular section with smooth zero locus $\mathcal{Z} := Z(f) = Z(dw)$. Note that this includes the special case of $V_2 = 0$.

(c) *A GLSM is Calabi–Yau if $\mathcal{T}$ is Calabi–Yau.*

Example 1.4.6. Consider the case where $V = V_1$ i.e. $V_2 = 0$ where V is a G-representation such that $(\mathrm{Sym}V^\vee)^G = \mathbb{C}$. We equip this with a trivial $\mathbb{C}^*_R$-action and set $\Gamma = G \times \mathbb{C}^*_R$. View $w = 0$ as a section of $\mathcal{O}(\chi)$ where χ is the projection character. Take a generic stability condition θ so that the GIT stack quotient $[V/\!\!/_\theta G]$ is a smooth proper Deligne–Mumford stack. This can be thought of as the GLSM corresponding to $[V/\!\!/_\theta G]$ itself.

The following special case gives the GLSM theory for the quintic 3-fold, it is simply a specific example of a geometric phase. In general, geometric phases can be used to create GLSM theories for zero loci of regular sections of homogeneous vector bundles in GIT quotients.

Example 1.4.7. Consider the vector space
$$V = V_1 \oplus V_2 = \mathbb{C}^5 \times \mathbb{C} = \mathrm{Spec}(\mathbb{C}[x_1,\dots,x_5,p]),$$
with an action of $G = \mathbb{C}^*$ with weights $(1,1,1,1,1,-5)$. The superpotential $w\colon V \to \mathbb{C}$ is given by
$$w = p\sum_{i=1}^5 x_i^5.$$
This function is homogeneous of degree 1 if we choose our R-charge action to have weights $(0,0,0,0,0,1)$. If we choose θ to be the identity character, then $\mathcal{T} \to \mathcal{X}$ is given by $\mathrm{tot}(\mathcal{O}_{\mathbb{P}^4}(-5)) \to \mathbb{P}^4$ and $\mathcal{Z}$ is the quintic threefold $\{\sum_{i=1}^5 x_i^5 = 0\} \subseteq \mathbb{P}^4$.

Changing the stability condition in the previous example flips the situation from Gromov–Witten theory to FJRW theory.

Example 1.4.8. Let V, G and w be as in the previous example, but modify the R-charge to have weights $(1,1,1,1,1,0)$ and let $\theta = -\mathrm{Id}$. Now w is homogeneous of degree 5 with respect to the R-charge grading. Re-ordering our decomposition of $V = V_1 \oplus V_2$, we choose $V_1 = \mathrm{Spec}(\mathbb{C}[p])$ and $V_2 = \mathrm{Spec}(\mathbb{C}[x_1, \dots, x_5])$. Then $V /\!\!/_\theta G \to \mathcal{X}$ is given by $[\mathbb{C}^5/\mu_5] \to B\mu_5$, and when descended to the quotient, the potential becomes the function $w\colon [\mathbb{C}^5/\mu_5] \to \mathbb{C}$ given by the Fermat polynomial $\sum_{i=1}^5 x_i^5$. Note that the nondegeneracy condition in this case reduces to the requirement that the descended w has an isolated singularity at the origin. In this case we prove that the GLSM invariants we construct agree with Polishchuck and Vaintrob's construction of FJRW invariants (see §6.2 and 6.3).

The previous example can also be achieved more simply.

Example 1.4.9. Consider the case where $V = V_2$ i.e. $V_1 = 0$, $\mathbb{C}^*_R$ acts with positive weights, and w is a quasi-homogeneous polynomial with respect to these weights with an isolated singularity at the origin (this is the nondegeneracy condition). Let G be any finite diagonal group of symmetries of w. This is the general setup of FJRW theory.

For completeness we also give an example where V_2 is not a sum of one-dimensional representations.

Example 1.4.10. Let $W = \mathbb{C}^3$, $G = \mathrm{Gl}(W)$, $V_1 = \mathrm{Hom}(W, \mathbb{C}^6)$, and $V_2 = W \otimes \det W \oplus \bigwedge^2 W$. Let θ be the determinant character and $\mathbb{C}^*_R$ act trivially on V_1 and by scaling on V_2. Denote by S the tautological subbundle on $\mathrm{Gr}(3,6) = V_1 /\!\!/_\theta G$. We take an element $w \in ((\mathrm{Sym}\, V_1^\vee) \otimes V_2^\vee)^G$, general enough so that the corresponding section σ of $S^*(1) \oplus \bigwedge^2 S^*$ defines a codimension 6, smooth zero locus $Z(\sigma)$ in $\mathrm{Gr}(3,6)$. This is a Calabi-Yau 3-fold. For more such examples, see e.g., Table 1 of [IIM16].

Remark 1.4.11. Note that in the previous example, $Z(\sigma)$ in $\mathrm{Gr}(3,6)$ coincides with $Z(dw)$ in $[V_1 \oplus V_2 /\!\!/_\theta G]$. This is the case for any geometric phase.

1.5. **The plan.** Even though Landau–Ginzburg maps generally land in the stack $[V^{ss}(\nu)/\Gamma]$, there are evaluation maps at the markings

$$\mathrm{ev}^i : LG_{g,r}(\mathcal{T}, d) \to I\mathcal{T}$$

to the inertia stack of the GIT quotient $\mathcal{T} := [V /\!\!/_\theta G]$. The most important structure needed to define the A-model of a nondegenerate GLSM is a virtual class in the homology of the proper moduli space $LG_{g,r}(\mathcal{Z}, d)$ of stable LG maps to the critical locus $\mathcal{Z} := Z(dw)$. In [FJR17], such a class is constructed algebraically, via the cosection localization method of Kiem and Li [KL13], only under certain restrictions; essentially when the evaluation maps are required to land in a proper substack of $I\mathcal{T}$, see [FJR17, Def. 6.1.6]. As a consequence, these virtual classes are in general not sufficient to produce the desired outcome of the construction of a Cohomological Field Theory (CohFT) in the sense of Kontsevich and Manin [KM94]. (In the parlance of [FJR13, FJR17], splitting at a *broad node* cannot be handled using the cosection localized virtual classes.)

The ultimate goal of our project is to give a full algebraic construction of a CohFT for GLSM targets by generalizing the approach of Polishchuck and Vaintrob, [PV16], from the FJRW-theory of hypersurface singularities [FJR13]. The idea of this approach is to "lift" the homological virtual class to an object in an appropriate derived category of factorizations. The CohFT is then obtained by performing a Fourier–Mukai transform and passing to Hochschild homology.

Specifically, we seek to implement the following program. Let $(V, G, \mathbb{C}^*_R, \theta, \nu, w)$ be a nondegenerate GLSM target as above. Fix $g, r \geq 0$ in the stable range, i.e., satisfying $2g - 2 + r > 0$, and choose $d \in \mathrm{Hom}_{\mathbb{Z}}(\widehat{G}, \mathbb{Q})$. Consider the diagram

$$\begin{tikzcd} LG_{g,r}(\mathcal{Z}, d) \arrow[r, hook] & LG_{g,r}(\mathcal{T}, d) \arrow[dl, "\mathrm{ev}"'] \arrow[dr, "\mathrm{st}"] & \\ (I\mathcal{T})^r & & \overline{\mathcal{M}}_{g,r} \end{tikzcd}$$

with $\mathrm{ev} = (\mathrm{ev}_1, \ldots, \mathrm{ev}_r)$ and st the stabilization map which forgets the data $(\mathcal{P}, \sigma, \varkappa)$ and stabilizes the domain curve after removing orbifold structures on the markings. Then there is an appropriate extension of this diagram as follows:

$$\begin{tikzcd} & & \mathcal{O}_{\square} \xrightarrow{\beta} E \xrightarrow{\alpha} \mathcal{O}_{\square} \otimes \mathbb{C}(\chi) \arrow[d, bend left] & \\ LG_{g,r}(\mathcal{Z}, d) \arrow[r, hook] & LG_{g,r}(\mathcal{T}, d) \arrow[r, hook] & \square \arrow[dl, "\mathrm{ev}"'] \arrow[dr, "\mathrm{st}"] & \\ & (I\mathcal{T})^r & & \overline{\mathcal{M}}_{g,r} \end{tikzcd}$$

where in §4 we construct a stack $\square = \square_{g,r,d}$ with a $\mathbb{C}^*_R$-action, a $\mathbb{C}^*_R$-equivariant vector bundle $E \to \square$, and $\mathbb{C}^*_R$-equivariant sections $\alpha \in \Gamma(E^\vee \otimes \mathbb{C}(\chi)) = \mathrm{Hom}(E, \mathcal{O}_\square \otimes \mathbb{C}(\chi))$, $\beta \in \Gamma(E) = \mathrm{Hom}(\mathcal{O}_\square, E)$ satisfying the following properties:

(1) The space $\square$ is a smooth separated Deligne–Mumford stack with a "stabilization map" $\mathrm{st} : \square \to \overline{\mathcal{M}}_{g,r}$ and with a pure dimension in each twisted component given by the formula

$$\dim H^0(\mathcal{V}|_{C_s}) - \dim H^1(\mathcal{V}|_{C_s}) + \dim \mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma)_{\omega^{\log}_{\mathfrak{C}}} + \operatorname{rank} E$$

(see §3.1.1 for the definition of the Artin stack $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma)_{\omega^{\log}_{\mathfrak{C}}}$).

(2) For $1 \leq i \leq r$ there exist smooth evaluation maps $\mathrm{ev}^i \colon \square \to I\mathcal{T}$ which are $\mathbb{C}^*_R$-equivariant.

(3) (a) The vanishing locus of β is canonically isomorphic to $LG_{g,r}(\mathcal{T}, d)$,
(b) the common vanishing locus of α and β is canonically isomorphic with $LG_{g,r}(\mathcal{Z}, d)$ (see Proposition 2.3.3 and 5.1.5) for which ev^i maps and st maps are compatible, and
(c) the composition $-\alpha^\vee \circ \beta$ is equal to $-(\boxplus_{i=1}^r \mathrm{ev}^i)^*(w)$.

Note that in (3.c) above, we endow $I\mathcal{T}$ with a superpotential, which we also call w by abuse of notation, via composition with the natural morphism $I\mathcal{T} \to \mathcal{T}$. Given the above, we define the Koszul factorization

$$K_{g,r,d} := \{-\alpha, \beta\} \in \mathrm{D}([\square/\mathbb{C}^*_R], -\mathrm{ev}^*(\boxplus_{i=1}^r w))$$

which is supported on $LG_{g,n}(\mathcal{Z}, d)$. We call this the *fundamental factorization*. It will play the role of the virtual fundamental class for $LG_{g,n}(\mathcal{Z}, d)$. Namely, $K_{g,r,d}$ defines a Fourier–Mukai transform $\Phi_{K_{g,r,d}}$ for categories of factorizations,

$$\begin{tikzcd} & \mathrm{D}([\square/\mathbb{C}^*_R], \mathrm{ev}^*(\boxplus_{i=1}^r w)) \arrow[r, "\overset{\mathbb{L}}{\otimes} K_{g,r,d}"] & \mathrm{D}([\square/\mathbb{C}^*_R], 0)_{LG_{g,n}(\mathcal{Z},d)} \arrow[dr, "\mathbb{R}\,\mathrm{st}_*"] & \\ \prod_{i=1}^r \mathrm{D}(I[V^{ss}(\theta)/\Gamma], w) \arrow[ur, "\mathbb{L}\,\mathrm{ev}^*"] & & & \mathrm{D}(\overline{\mathcal{M}}_{g,r}) \end{tikzcd}$$

i.e.

$$\Phi_{K_{g,r,d}}(\mathcal{E}) := \mathbb{R}\,\mathrm{st}_*(\mathbb{L}\,\mathrm{ev}^*\,\mathcal{E} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_\square} K_{g,r,d}).$$

The induced map on Hochschild homology (after being suitably adjusted by an appropriate Todd class modification), intermixed with an HKR morphism from Hochschild homology to cohomology gives us a collection of invariants (see §5.5 the precise formulation). In a sequel [CFG$^+$] we show that the set of these invariants for all g, r, d, form a cohomological field theory.

Remark 1.5.1. The construction is in fact a bit more involved: in order to correct the invariants by an appropriate Todd class as in [PV16, Equation (5.15)], we factorize the map st as $\square \to \widetilde{U} \to \overline{\mathcal{M}}_{g,r}$ where $\widetilde{U}$ is a proper and smooth Deligne–Mumford stack on which we multiply by the Todd correction (precisely, see Definition 5.5.1).

2. Factorizations

Factorizations are natural objects attached to an LG model, in the same way complexes of coherent sheaves are natural objects attached to a variety. In this section, we recall the definition of the derived category of factorizations and some properties that will be use in the paper. The true heart of our work will be to construct a natural factorization, called the fundamental factorization (see Definition 5.1.4), for each moduli space of LG stable maps (under the condition it is a convex Hybrid Model, see §4.1). The fundamental factorization plays a similar role to the virtual structure sheaf in Gromov–Witten theory.

2.1. **Derived categories of Landau–Ginzburg models.**

In order to naturally define derived functors, we use the more sheaf-theoretic approach to factorizations introduced by Lin–Polmerleano and Positselski [LP13, EP15]. We review the necessary concepts below.

Let $\mathcal{X}$ be an algebraic stack. equipped with a line bundle $\mathcal{L}$ and a section w of $\mathcal{L}$. A factorization is the data $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_0, \phi_1^{\mathcal{E}}, \phi_0^{\mathcal{E}})$ where $\mathcal{E}_1$, $\mathcal{E}_0$ are quasi-coherent sheaves on $\mathcal{X}$ and

$$\mathcal{E}_1 \xrightarrow{\phi_1^{\mathcal{E}}} \mathcal{E}_0 \xrightarrow{\phi_0^{\mathcal{E}}} \mathcal{E}_1 \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}$$

are morphisms such that

$$\phi_0^{\mathcal{E}} \circ \phi_1^{\mathcal{E}} = w,$$
$$(\phi_1^{\mathcal{E}} \otimes 1_{\mathcal{L}}) \circ \phi_0^{\mathcal{E}} = w.$$

Definition 2.1.1. *The shift, denoted by* $[1]$*, sends a factorization,* $\mathcal{E}$*, to the factorization,*

$$\mathcal{E}[1] := (\mathcal{E}_0, \mathcal{E}_1 \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}, -\phi_0^{\mathcal{E}}, -\phi_1^{\mathcal{E}} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}).$$

A morphism between two factorizations of even degree $f\colon \mathcal{E} \to \mathcal{F}[2k]$ is a pair $f = (f_0, f_1)$ defined by

$$\mathrm{Hom}^{2k}_{\mathsf{Fact}(\mathcal{X},w)}(\mathcal{E},\mathcal{F}) := \mathrm{Hom}_{\mathrm{Qcoh}(\mathcal{X})}(\mathcal{E}_0, \mathcal{F}_0 \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}^k) \oplus \mathrm{Hom}_{\mathrm{Qcoh}(\mathcal{X})}(\mathcal{E}_1, \mathcal{F}_0 \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}^k)$$

and, similarly, a morphism of odd degree $f\colon \mathcal{E} \to \mathcal{F}[2k+1]$ is a pair $f = (f_0, f_1)$ defined by

$$\mathrm{Hom}^{2k+1}_{\mathsf{Fact}(\mathcal{X},w)}(\mathcal{E},\mathcal{F}) := \mathrm{Hom}_{\mathrm{Qcoh}(\mathcal{X})}(\mathcal{E}_0, \mathcal{F}_1 \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}^{k+1}) \oplus \mathrm{Hom}_{\mathrm{Qcoh}(\mathcal{X})}(\mathcal{E}_1, \mathcal{F}_0 \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}^k).$$

You can equip these Hom sets with a differential coming from the graded commutator with the morphisms defining $\mathcal{E}$ and $\mathcal{F}$. This yields a $\mathbb{Z}$-graded dg category $\mathsf{Fact}(\mathcal{X}, w)$.

Denote by $Z^0\mathsf{Fact}(\mathcal{X}, w)$ the subcategory of $\mathsf{Fact}(\mathcal{X}, w)$ with the same objects but whose morphisms are only the closed degree zero morphisms between any two objects $\mathcal{E}$ and $\mathcal{F}$.

Remark 2.1.2. The morphisms in $Z^0\mathsf{Fact}(\mathcal{X}, w)$ are just the commuting morphisms of factorizations.

The category $Z^0\mathsf{Fact}(\mathcal{X}, w)$ is abelian. Hence, the notion of a complex of objects in $Z^0\mathsf{Fact}(\mathcal{X}, w)$ makes sense.

Definition 2.1.3. *Given a complex of objects*

$$\ldots \to \mathcal{E}^b \xrightarrow{f^b} \mathcal{E}^{b+1} \xrightarrow{f^{b+1}} \ldots$$

in $Z^0\mathsf{Fact}(\mathcal{X}, w)$, we define the **totalization** *of the complex to be the factorization $\mathcal{T} \in \mathsf{Fact}(\mathcal{X}, w)$ given by the data:*

$$\mathcal{T}_0 := \bigoplus_{i=2l} \mathcal{E}_0^i \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}^{-l} \oplus \bigoplus_{i=2l+1} \mathcal{E}_1^i \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}^{-l}$$

$$\mathcal{T}_1 := \bigoplus_{i=2l} \mathcal{E}_1^i \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}^{-l} \oplus \bigoplus_{i=2l-1} \mathcal{E}_0^i \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}^{-l}$$

$$\phi_0^{\mathcal{T}} := \begin{pmatrix} \ddots & 0 & 0 & 0 & 0 \\ \ddots & -\phi_1^{\mathcal{E}^1} & 0 & 0 & 0 \\ 0 & f_0^0 & \phi_0^{\mathcal{E}^0} & 0 & 0 \\ 0 & 0 & f_1^{-1} & -\phi_1^{\mathcal{E}^1} \otimes \mathcal{L}^{-1} & 0 \\ 0 & 0 & 0 & \ddots & \ddots \end{pmatrix}$$

$$\phi_1^{\mathcal{T}} := \begin{pmatrix} \ddots & 0 & 0 & 0 & 0 \\ \ddots & -\phi_0^{\mathcal{E}^1} \otimes \mathcal{L} & 0 & 0 & 0 \\ 0 & f_1^1 \otimes \mathcal{L} & \phi_1^{\mathcal{E}^0} & 0 & 0 \\ 0 & 0 & f_0^0 & -\phi_0^{\mathcal{E}^1} & 0 \\ 0 & 0 & 0 & \ddots & \ddots \end{pmatrix}$$

Denote by $\mathsf{Acyc}(\mathcal{X}, w)$ the full saturated subcategory of $\mathsf{Fact}(\mathcal{X}, w)$ consisting of totalizations of bounded exact complexes of $Z^0\mathsf{Fact}(\mathcal{X}, w)$. Objects of $\mathsf{Acyc}(\mathcal{X}, w)$ are called $\mathsf{acyclic}$. Also, denote by $[\mathsf{C}]$ the homotopy category of a small k-linear dg category C.

We have the following general definition.

Definition 2.1.4. *The* ***absolute derived category*** $\mathrm{D}^{abs}[\mathsf{Fact}(\mathcal{X}, w)]$ *of* $[\mathsf{Fact}(\mathcal{X}, w)]$ *is the Verdier quotient of* $[\mathsf{Fact}(\mathcal{X}, w)]$ *by* $[\mathsf{Acyc}(\mathcal{X}, w)]$*. The* ***derived category*** *of* $(\mathcal{X}, w)$ *is the full subcategory of* $\mathrm{D}^{abs}[\mathsf{Fact}(\mathcal{X}, w)]$ *generated by factorizations with coherent components. We denote this category by the abbreviated notation* $\mathrm{D}(\mathcal{X}, w)$*.*

Remark 2.1.5. As the notation suggests, the category $\mathrm{D}(\mathcal{X}, w)$ can be thought of as the derived category of the gauged Landau–Ginzburg model $(\mathcal{X}, w)$.

2.2. **Derived functors.** In this section, the goal is to define Fourier–Mukai functors in the derived category of factorizations. They are of prime interest in this paper, since they enter explicitly into the definition of our GLSM theory. We start with the definitions of derived pullbacks, pushforwards, and tensor products.

Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of algebraic stacks. From the section w of $\mathcal{L}$ on $\mathcal{X}$, we can pullback to get a section f^*w of $f^*\mathcal{L}$ on $\mathcal{Y}$. At the level of dg categories, we get the following functors:

Definition 2.2.1.

$$\begin{aligned} f^*\colon \mathsf{Fact}(\mathcal{X}, w) &\to \mathsf{Fact}(\mathcal{Y}, f^*w) \\ (\mathcal{E}_1, \mathcal{E}_0, \phi_1^{\mathcal{E}}, \phi_0^{\mathcal{E}}) &\mapsto (f^*\mathcal{E}_1, f^*\mathcal{E}_0, f^*\phi_1^{\mathcal{E}}, f^*\phi_0^{\mathcal{E}}) \end{aligned}$$

and

$$\begin{aligned} f_*\colon \mathsf{Fact}(\mathcal{Y}, f^*w) &\to \mathsf{Fact}(\mathcal{X}, w) \\ (\mathcal{F}_1, \mathcal{F}_0, \phi_1^{\mathcal{F}}, \phi_0^{\mathcal{F}}) &\mapsto (f_*\mathcal{F}_1, f_*\mathcal{F}_0, f_*\phi_1^{\mathcal{F}}, f_*\phi_0^{\mathcal{F}}). \end{aligned}$$

Note that by the projection formula $f_*(\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} f^*\mathcal{L}) \cong (f_*\mathcal{F}) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}$ *under which* $f_*(f^*w)$ *corresponds to* w *so this is well-defined. Let* v *be a section of* $\mathcal{L}$ *on* $\mathcal{X}$*. We define a dg-functor,*

$$\otimes_{\mathcal{O}_\mathcal{X}}\colon \mathsf{Fact}(\mathcal{X}, w) \otimes_k \mathsf{Fact}(\mathcal{X}, v) \to \mathsf{Fact}(\mathcal{X}, v + w)$$

by setting

$$\begin{aligned} (\mathcal{E} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{F})_0 &:= \mathcal{E}_0 \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{F}_0 \oplus \mathcal{E}_1 \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{F}_1 \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L} \\ (\mathcal{E} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{F})_1 &:= \mathcal{E}_0 \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{F}_1 \oplus \mathcal{E}_1 \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{F}_0 \\ \phi_0^{\mathcal{E} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{F}} &:= \begin{pmatrix} 1_{\mathcal{E}_0} \otimes_{\mathcal{O}_\mathcal{X}} \phi_0^{\mathcal{F}} & \phi_1^{\mathcal{E}} \otimes_{\mathcal{O}_\mathcal{X}} 1_{\mathcal{F}_1} \\ -\phi_0^{\mathcal{E}} \otimes_{\mathcal{O}_\mathcal{X}} 1_{\mathcal{F}_0} & 1_{\mathcal{E}_1} \otimes_{\mathcal{O}_\mathcal{X}} \phi_1^{\mathcal{F}} \end{pmatrix} \\ \phi_1^{\mathcal{E} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{F}} &:= \begin{pmatrix} 1_{\mathcal{E}_0} \otimes_{\mathcal{O}_\mathcal{X}} \phi_1^{\mathcal{F}} & -\phi_1^{\mathcal{E}} \otimes_{\mathcal{O}_\mathcal{X}} 1_{\mathcal{F}_0} \\ \phi_0^{\mathcal{E}} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L} \otimes_{\mathcal{O}_\mathcal{X}} 1_{\mathcal{F}_1} & 1_{\mathcal{E}_1} \otimes_{\mathcal{O}_\mathcal{X}} \phi_0^{\mathcal{F}} \end{pmatrix} \end{aligned}$$

Let $\mathrm{D}^{\mathrm{abs}}[\mathrm{Fact}(\mathcal{X}, w)]_{f^*}$ denote the full subcategory of $\mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathcal{X}, w)]$ consisting of factorizations with f^*-acyclic components. Similarly, let $\mathrm{D}^{\mathrm{abs}}[\mathrm{Fact}(\mathcal{X}, w)]_{f_*}$ denote the full subcategory of $\mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathcal{X}, w)]$ consisting of factorizations with f_*-acyclic components and $\mathrm{D}^{\mathrm{abs}}[\mathrm{Fact}(\mathcal{X}, w)]_{\otimes}$ denote the full subcategory of $\mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathcal{X}, w)]$ consisting of factorizations with flat components.

Definition 2.2.2 ([PV11, §3.1])**.** *An algebraic stack* $\mathcal{X}$ *is called a* ***nice quotient stack*** *if* $\mathcal{X} = [T/H]$ *where* T *is a noetherian scheme and* H *is a reductive linear algebraic group such that* T *has an ample family of* H*-equivariant line bundles.*

From now on assume that $\mathcal{X}$ is a smooth nice quotient stack. Then, by [BDF+16, Proposition 2.20] the inclusions induce equivalences of categories

$$\mathrm{D}^{\mathrm{abs}}[\mathrm{Fact}(\mathcal{X},w)]_{f^*} \xrightarrow{\iota_{f^*}} \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathcal{X},w)]$$
$$\mathrm{D}^{\mathrm{abs}}[\mathrm{Fact}(\mathcal{X},w)]_{\otimes} \xrightarrow{\iota_{\otimes}} \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathcal{X},w)].$$

Definition 2.2.3. *The derived pullback is the functor*

$$\mathbb{L}f^*\colon \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathcal{X},w)] \to \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathcal{Y},f^*w)]$$
$$\mathcal{E} \mapsto f^*\iota_{f^*}^{-1}\mathcal{E}.$$

Let $\mathcal{F} \in \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathcal{X},v)]$. The derived tensor product is the functor

$$(- \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_\mathcal{X}} \mathcal{F})\colon \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathcal{X},w)] \to \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathcal{X},w+v)]$$
$$\mathcal{E} \mapsto \iota_{\otimes}^{-1}\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{E}.$$

Definition 2.2.4. *Let $\mathcal{M}$ be a closed substack of $\mathcal{X}$. Then, by definition, the map $\mathcal{M} \to \mathcal{X}$ is representable. We say that a factorization $\mathcal{E}$ has support on $\mathcal{M}$ if for any scheme T and any morphism $T \to \mathcal{X}$ the restriction of $\mathcal{E}$ to $(T \times_\mathcal{X} \mathcal{X})\backslash(T \times_\mathcal{X} \mathcal{M})$ is acyclic. We denote the full subcategory of $\mathrm{D}(\mathcal{X},w)$ consisting of factorizations with support on $\mathcal{M}$ by $\mathrm{D}(\mathcal{X},w)_\mathcal{M}$.*

Our goal now is to similarly define $\mathbb{R}f_*$ and show that it restricts to a functor from the properly supported objects of $\mathrm{D}(\mathcal{Y},f^*w)$ to $\mathrm{D}(\mathcal{X},w)$. We will require this in the following setup.

Let $\mathcal{X},\mathcal{Y}$ be smooth nice quotient stacks, w be a section of the line bundle $\mathcal{Y}$, and $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of finite homological dimension.

Consider a smooth affine atlas

$$p_0\colon Y_0 \to \mathcal{Y}.$$

Following Nironi [Nir08], we can define a f_*-acyclic resolution of any coherent sheaf $\mathcal{F}$ on $\mathcal{Y}$ by

$$0 \to \mathcal{F} \to p_{0*}p_0^*\mathcal{F} \to p_{0*}p_0^*K_1 \otimes \mathcal{F} \to ...$$

where K_i is the cokernel of the map $K_{i-1} \to p_{0*}p_0^*K_{i-1}$ with initial setting $K_0 := \mathcal{O}_\mathcal{Y}$. Denote by

$$\check{\mathcal{C}} := 0 \to p_{0*}p_0^*\mathcal{O}_\mathcal{X} \to p_{0*}p_0^*K_1 \to ... \tag{2.1}$$

the above resolution of $\mathcal{O}_\mathcal{Y}$ and notice that, by the projection formula, in general, the resolution of $\mathcal{F}$ can be rewritten as $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{Y}} \check{\mathcal{C}}$. Let $\check{\mathcal{C}}_{\leq n}$ be the good truncation of $\check{\mathcal{C}}_{\leq n}$ at n. Notice that, given a factorization, $\mathcal{E} \in \mathrm{D}(\mathcal{Y},f^*w)$, we have an isomorphism in $\mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathcal{Y},f^*w)]$

$$\mathcal{E} \cong \mathcal{E} \otimes_{\mathcal{O}_\mathcal{Y}} \check{\mathcal{C}}_{\leq n}$$

where the right hand side is the totalization of the complex of factorizations

$$... \to \mathcal{E} \otimes_{\mathcal{O}_\mathcal{Y}} \check{\mathcal{C}}_{i-1} \to \mathcal{E} \otimes_{\mathcal{O}_\mathcal{Y}} \check{\mathcal{C}}_i \to \mathcal{E} \otimes_{\mathcal{O}_\mathcal{Y}} \check{\mathcal{C}}_{i+1} \to ...$$

Remark 2.2.5. The complex $\check{\mathcal{C}}$ can be regarded as an analog of the Čech complex for algebraic stacks.

Following [EP15, Remark 1.3], the notion of acyclicity of a factorization is local for the smooth topology.

Proposition 2.2.6. *If U is a smooth atlas of $\mathcal{X}$ and the pullback of $\mathcal{E}$ to U is acyclic, then $\mathcal{E}$ is acyclic.*

Proof. Using $\check{\mathcal{C}}$ from Equation (2.1), we have that

$$\mathcal{E} \cong \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{Y}}} \check{\mathcal{C}}$$

in the co-derived category of factorizations. Hence, $\mathcal{E}$ is the totalization of a complex of factorizations each of which is acyclic. Hence $\mathcal{E}$ is co-acyclic by definition. Since $\mathcal{E}$ is coherent, this implies it is acyclic, see e.g. [EP15, Proposition 1.5 (d)] or [Pos11, Theorem 1, Section 3.11]. □

Corollary 2.2.7. *The support of a coherent factorization $\mathcal{E}$ is local in the smooth topology.*

Proof. This follows immediately from the above proposition. □

Let $\mathcal{M} \subseteq \mathcal{Y}$ be a closed substack such that the composition

$$\mathcal{M} \to \mathcal{X}$$

is proper.

Proposition 2.2.8. *Assume that $f : \mathcal{Y} \to \mathcal{X}$ has finite cohomological dimension n. There is a well-defined derived pushforward functor*

$$\begin{aligned} \mathbb{R}f_*\colon \mathrm{D}(\mathcal{Y}, f^*w)_{\mathcal{M}} &\to \mathrm{D}(\mathcal{X}, w) \\ \mathcal{E} &\mapsto f_*(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{Y}}} \check{\mathcal{C}}). \end{aligned}$$

Proof. As $\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{Y}}} \check{\mathcal{C}_{\leq n}}$ is f_*-acyclic, this gives a functorial definition of a pushforward functor which naturally lands in the absolute derived category of quasi-coherent factorizations. We need to show that $f_*(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{Y}}} \check{\mathcal{C}}_{\leq n})$ actually lies in $\mathrm{D}(\mathcal{X}, w)$ i.e. that it is generated by factorizations with coherent components.

Let $\mathcal{E}_0, \mathcal{E}_1$ be the components of $\mathcal{E}$. First, suppose that these components are coherent and supported on $\mathcal{M}$. Now, notice that $f_*(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{Y}}} \check{\mathcal{C}}_{\leq n})$ can alternatively be described as the totalization of the complex of factorizations

$$D := 0 \to f_*(p_{0*}p_0^* \otimes E) \to f_*(\check{\mathcal{C}}_1 \otimes E) \to ... \to f_*(\check{\mathcal{C}}_{n-1} \otimes E) \to f_*(\mathrm{coker} \otimes E) \to 0$$

in the abelian category of factorizations.

Then, $\mathrm{H}^i(D)$ is a factorization whose components are

$$\mathrm{H}^i(f_*(\mathcal{E}_0 \otimes \check{\mathcal{C}}_{\leq n})) = \mathbb{R}^i f_* \mathcal{E}_0$$

and

$$\mathrm{H}^i(f_*(\mathcal{E}_1 \otimes \check{\mathcal{C}}_{\leq n})) = \mathbb{R}^i f_* \mathcal{E}_1.$$

Since, by assumption, $\mathcal{E}_0, \mathcal{E}_1$ are supported on $\mathcal{M}$ which is proper over $\mathcal{X}$, it follows that the components above are coherent. Therefore, $f_*(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{Y}}} \check{\mathcal{C}}_{\leq n})$ can be obtained by a sequence of exact triangles involving coherent factorizations, hence, lies in $\mathrm{D}(\mathcal{X}, w)$ as desired.

The result now follows from Lemma 2.2.9 as we have shown that a set of compact generators for $\mathrm{D}(\mathcal{Y}, f^*w)_{\mathcal{M}}$ land in $\mathrm{D}(\mathcal{X}, w)$. □

Lemma 2.2.9. *Assume that $f : \mathcal{Y} \to \mathcal{X}$ has finite cohomological dimension. The category $\mathrm{D}(\mathcal{Y}, f^*w)_{\mathcal{M}}$ is split generated by factorizations whose components are coherent and supported on $\mathcal{M}$.*

Proof. This is a repetition of the argument of [EP15, Corollary 1.10 (b)]. □

Proposition 2.2.10 (Projection Formula)**.** *Let $\mathcal{X}, \mathcal{Y}$ be smooth nice quotient stacks. Let w, v be sections of a line bundle $\mathcal{L}$ on $\mathcal{X}$. Suppose that*

$$f : \mathcal{Y} \to \mathcal{X}$$

is a morphism of smooth quotient stacks. For any $\mathcal{F} \in \mathrm{D}(\mathcal{X}, v)$ and $\mathcal{E} \in \mathrm{D}(\mathcal{Y}, f^*w)_{\mathcal{M}}$ one has an isomorphism in $\mathrm{D}(\mathcal{X}, w + v)$,

$$\mathbb{R}f_*(\mathcal{E} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{\mathcal{Y}}} \mathbb{L}f^*\mathcal{F}) = \mathbb{R}f_*(\mathcal{E}) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}.$$

Proof. Since we are working in derived categories on nice quotient stacks, we may assume that $\mathcal{F}$ is locally-free and that $\mathcal{E}$ has been replaced by $\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}}} \check{\mathcal{C}}$. Then, since the usual projection formula is functorial, we may apply it component-wise to our factorization and the morphisms between components. □

Definition 2.2.11. *Let $\mathcal{X}$ be a smooth nice quotient stack. Let $\mathcal{L}, \mathcal{N}$ be line bundles on $\mathcal{X}, \mathcal{Y}$ respectively and $w \in \mathrm{H}^0(\mathcal{X}, \mathcal{L}), v \in \mathrm{H}^0(\mathcal{Y}, \mathcal{N})$. Let*

$$(f, g) : \mathcal{Z} \to \mathcal{X} \times \mathcal{Y}$$

be an algebraic stack lying over $\mathcal{X} \times \mathcal{Y}$ and assume that $f^*\mathcal{L} \cong g^*\mathcal{N}$. Let $P \in \mathrm{D}(\mathcal{Z}, g^*v - f^*w)$ such that the support of P is proper over $\mathcal{Y}$. The Fourier–Mukai transform by P is the functor

$$\begin{aligned} \Phi_P : \mathrm{D}(\mathcal{X}, w) &\to \mathrm{D}(\mathcal{Y}, v) \\ \mathcal{E} &\mapsto \mathbb{R}g_*(P \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{\mathcal{Z}}} \mathbb{L}f^*\mathcal{E}) \end{aligned}$$

Remark 2.2.12. In §4.1, we construct a *fundamental factorization* for each moduli space of convex hybrid model LG maps. This factorization is used as a kernel for a Fourier–Mukai transform whose passage to Hochschild homology defines enumerative GLSM invariants. Our fundamental factorization is a particular Koszul factorization, a concept which we will now define.

2.3. **Koszul factorizations.** In [PV16], one of the main components of the construction is the so called Koszul factorization. This construction, in fact, goes all the way back to the birth of factorizations [Eis80].

Definition 2.3.1. *Let $\mathcal{E}$ be a locally-free sheaf of finite rank on $\mathcal{X}$. Suppose we have the following commutative diagram of morphisms,*

$$\begin{tikzcd} \mathcal{O}_{\mathcal{X}} \arrow[r, "\beta"] \arrow[rr, bend right, "w"'] & \mathcal{E}^\vee \arrow[r, "\alpha^\vee"] & \mathcal{L} \end{tikzcd}$$

i.e. $\alpha \in \mathrm{H}^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}), \beta \in \mathrm{H}^0(\mathcal{X}, \mathcal{E}^\vee)$ *and* $\alpha^\vee \circ \beta = w$. *The Koszul Factorization* $\{\alpha, \beta\}$ *associated to the data* $(\mathcal{E}, \alpha, \beta)$ *is defined as*

$$\{\alpha,\beta\}_1 := \bigoplus_{l\geq 0}(\bigwedge^{2l+1}\mathcal{E}) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}^l$$

$$\{\alpha,\beta\}_0 := \bigoplus_{l\geq 0}(\bigwedge^{2l}\mathcal{E}) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}^l$$

$$\phi_0^{\{\alpha,\beta\}}, \phi_1^{\{\alpha,\beta\}} := \bullet \lrcorner \beta + \bullet \wedge \alpha,$$

where $\bullet \lrcorner \beta, \bullet \wedge \alpha$ *are the contraction and wedge operators respectively.*

Remark 2.3.2. If β is a regular section of $\mathcal{E}^\vee$, then $\{\alpha, \beta\}$ is isomorphic in $\mathrm{D}(\mathcal{X}, w)$ to the factorization $(0, \mathcal{O}_{Z(\beta)}, 0, 0)$, i.e., it is a locally-free replacement of this factorization (see, e.g., [BFK14, Proposition 3.20]).

Proposition 2.3.3. *The Koszul factorization* $\{\alpha, \beta\}$ *is supported on* $Z(\alpha) \cap Z(\beta)$.

Proof. By Corollary 2.2.7, the assertion is local in the smooth topology. Hence, we may assume that $\mathcal{X} = U$ is a scheme and $B = \mathcal{O}_U^n$. In this case, $\alpha = \oplus_{i=1}^n \alpha_i, \beta = \oplus_{i=1}^n \beta_i$ and

$$\{\alpha, \beta\} = \bigotimes_{i=1\dots n}^{\mathbb{L}} \{\alpha_i, \beta_i\}.$$

As the tensor product of n acyclic factorizations is acyclic, we have reduced to the case $B = \mathcal{O}_U$. Assume α is non-vanishing. In this case the identity map for $\{\alpha, \beta\}$ has an explicit homotopy,

$$\begin{tikzcd}
\mathcal{O}_U \arrow[r, "\beta^\vee"] \arrow[d, "\mathrm{Id}"'] & \mathcal{O}_U \arrow[r, "\alpha"] \arrow[d, "\mathrm{Id}"'] \arrow[dl, "0"'] & \mathcal{O}_U \arrow[d, "\mathrm{Id}"] \arrow[dl, "\alpha^{-1}"'] \\
\mathcal{O}_U \arrow[r, "\beta^\vee"] & \mathcal{O}_U \arrow[r, "\alpha"] & \mathcal{O}_U .
\end{tikzcd}$$

There is a similar explicit homotopy when β is non-vanishing. $\square$

2.4. **Hochschild Homology.** Hochschild homology takes the place of the cohomology of a variety when dealing with derived categories of factorizations, so that it will be the main ingredient for the state space of the GLSM theory.

Let C be a k-linear small dg-category, and let $\mathsf{C}(k)$ denotes the dg category of unbounded complexes over k. We have the diagonal C-C bimodule

$$\begin{aligned} \Delta_\mathsf{C} \colon \mathsf{C}^{\mathrm{op}} \otimes \mathsf{C} &\to \mathsf{C}(k) \\ (c, c') &\mapsto \mathrm{Hom}_\mathsf{C}(c, c'). \end{aligned}$$

This induces the trace functor

$$\mathrm{D}(\mathsf{C}^{\mathrm{op}} \otimes \mathsf{C}) \to \mathrm{D}(k), \quad F \mapsto F \overset{\mathbb{L}}{\otimes}_{\mathsf{C} \otimes \mathsf{C}^{\mathrm{op}}} \Delta_\mathsf{C},$$

where $\mathrm{D}(\mathsf{A})$ is the derived category of a k-linear dg-category A (see [PV12, §1.1]).

Definition 2.4.1. *The Hochschild homology* $\mathrm{HH}_*(\mathsf{C})$ *of* C *is defined to be the homology*

$$H^{-*}(\Delta_{\mathsf{C}} \overset{\mathbb{L}}{\otimes}_{\mathsf{C}^{\mathrm{op}}\otimes\mathsf{C}} \Delta_{\mathsf{C}}).$$

We denote by $\mathrm{HH}_*(\mathcal{X}, w)$ *the Hochschild homology of the category of factorizations with injective components whose isomorphism class lies in* $\mathrm{D}(\mathcal{X}, w)$ *(this is a natural dg enhancement of* $\mathrm{D}(\mathcal{X}, w)$*).*

Proposition 2.4.2. *Let* C *and* D *be saturated small dg-categories over* k*. Let* F *be an object of the dg-category* $\mathrm{perf}(\mathsf{C}^{\mathrm{op}} \otimes \mathsf{D})$ *of homotopically finitely presented right* $\mathsf{C}^{\mathrm{op}} \otimes \mathsf{D}$*-modules. Then, there is a homomorphism of vector spaces,*

$$F_*\colon \mathrm{HH}_*(\mathsf{C}) \to \mathrm{HH}_*(\mathsf{D}).$$

Moreover, the assignment, $F \mapsto F_*$*, is natural in the following sense. Let* $F_1 \in \mathrm{perf}(\mathsf{B}^{\mathrm{op}} \otimes \mathsf{C})$ *and* $F_2 \in \mathrm{perf}(\mathsf{C}^{\mathrm{op}} \otimes \mathsf{D})$ *and let* $F_2 \circ F_1$ *denote the right* B-D *bimodule corresponding to the tensor product* $F_1 \overset{\mathbb{L}}{\otimes}_{\mathsf{C}} F_2$*. Then,* $(F_2 \circ F_1)_* \cong F_{2*} \circ F_{1*}$*.*

Proof. This is [PV12, Lemma 1.2.1]. □

Definition 2.4.3. *Let* C *and* D *be saturated small dg-categories over* k*. Let* F *be an object of* $\mathrm{perf}(\mathsf{C}^{\mathrm{op}} \otimes_k \mathsf{D})$*. We will call the linear map,* F_**, the pushforward by* F*. In particular, for an object* $E \in \mathrm{perf}(\mathsf{C}) = \mathrm{perf}(k \otimes \mathsf{C})$*, we get an induced map,*

$$E_*\colon k \cong \mathrm{HH}_*(k) \to \mathrm{HH}_*(\mathsf{C}).$$

Then define,

$$\mathrm{ch}(E) := E_*(1)$$

and call it the Chern character of E*.*

Theorem 2.4.4 (Hochschild–Kostant–Rosenberg)**.** *Let* X *be a smooth projective variety. There is a natural isomorphism,*

$$\phi_{\mathrm{HKR}}\colon \bigoplus_l \mathrm{HH}_l(X) \to \bigoplus_l \bigoplus_{q-p=l} \mathrm{H}^p(X, \Omega_X^q).$$

such that the Chern character and classical Chern character agree under the HKR isomorphism,

$$\phi_{\mathrm{HKR}}(\mathrm{ch}(\mathcal{E})) = \mathrm{ch}(\mathcal{E}) \in \mathrm{H}^*(X)$$

where the left hand side is the dg Chern character and the right hand side is the usual chern character.

Proof. The HKR isomorphism in the affine case is due to [HKR09]. In this generality, it is due to Swan [Swa96, Corollary 2.6] and Kontsevich [Kon03], see also [Yek03]. The preservation of the Chern character was stated in [Mar01] and proven as [Căl05, Theorem 4.5].

□

Definition 2.4.5. *The isomorphism* ϕ_{HKR} *is called the Hochschild–Kostant–Rosenberg isomorphism or HKR isomorphism for short. Let* $\mathcal{Z}$ *be a smooth projective Deligne–Mumford stack realizable as a quotient stack. By* [Kre09, Proposition 5.1]*, there exists a finite flat surjective morphism* $\pi : U \to \mathcal{Z}$ *such that* U *is a scheme. Let* U *be a smooth projective variety with a finite flat surjective representable morphism*

$$\pi\colon U \to \mathcal{Z}.$$

We define the HKR morphism as

$$\bar{\phi}_{\mathrm{HKR}}\colon \mathrm{HH}_*(\mathcal{Z}) \xrightarrow{(\mathbb{L}\pi^*)_*} \mathrm{HH}_*(U) \xrightarrow{\phi_{\mathrm{HKR}}} \oplus \mathrm{H}^p(U, \Omega_U^q) \xrightarrow{\frac{1}{\deg \pi}\pi_*} \oplus \mathrm{H}^p(\mathcal{Z}, \Omega_{\mathcal{Z}}^q).$$

Remark 2.4.6. The definition of the HKR morphism is independent of the choice of U since given two finite surjective representable morphisms from U, V, we may pass to their fiber product. It then suffices to check that the HKR map for U, V agree with the one provided by $U \times_{\mathcal{Z}} V$. This follows from the fact that $r_* r^*$ is, cohomologically, multiplication by $\deg r$ (see, e.g., [Ful13] 1.7.4).

Remark 2.4.7. This definition follows the one in [PV16] but is less than optimal. In general, we anticipate an "HKR isomorphism" between the Hochschild homology of a smooth projective Deligne–Mumford stack and its Chen–Ruan cohomology. However, at present, this has only been proved in the global quotient case by a finite group [ACH14]. The map defined above should agree with the projection to the untwisted component of the inertia stack. This can be shown in the case where $\mathcal{Z}$ is a global quotient stack by a finite group.

Definition 2.4.8. *Let $\psi \in \mathrm{H}^{p,q}(X)$ be a cycle and $\mathcal{Z}$ be a smooth Deligne–Mumford stack with a representable morphism to $X \times \mathcal{Y}$. Let $\mathcal{K} \in \mathrm{D}(\mathcal{Z})$ be any object. Then, we define*

$$\Phi_{\mathcal{K}}^{\mathrm{HKR}}(\psi) := \bar{\phi}_{\mathrm{HKR}} \Phi_{\mathcal{K}} \phi_{\mathrm{HKR}}^{-1}(\psi).$$

Definition 2.4.9. *Given a cycle $\phi \in \mathrm{H}^*(\mathcal{Z})$ we define the cohomological integral transform to be*

$$\Phi_{\phi}^{\mathrm{H}} := g_*(f^*\psi \cup \phi).$$

where $f : \mathcal{Z} \to X$, $g : \mathcal{Z} \to \mathcal{Y}$ are the morphisms obtained from the projections.

2.5. **Comparisons.** Given a geometric phase GLSM, we compare in §6 the GLSM invariants with those of Gromov–Witten theory. The following general results will be useful for that purpose.

2.5.1. *Integral transforms and HKR.* The following theorem compares two induced Fourier–Mukai functors, one in Hochschild homology and the other in cohomology, under the HKR isomorphism.

Theorem 2.5.1 (Ramadoss)**.** *Let X be a smooth proper algebraic variety and $\mathcal{Y}$ be a smooth Deligne–Mumford global quotient stack with projective coarse moduli space. Let $\mathcal{Z}$ be a Deligne–Mumford stack with a representable morphism to $X \times \mathcal{Y}$ and $\mathcal{K}$ be an object of $\mathrm{D}(\mathcal{Z})$ which has proper support over $\mathcal{Y}$. For the integral transform, $\Phi_{\mathcal{K}}\colon \mathrm{D}(X) \to \mathrm{D}(\mathcal{Y})$, the action of $\Phi_{\mathcal{K}*}$ under the HKR morphism is equal to the cohomological integral transform associated to $\mathrm{td}(\mathcal{Z}/\mathcal{Y})\,\mathrm{ch}(\mathcal{K}) \in \mathrm{H}^\bullet(\mathcal{Z};\mathbb{C})$ i.e.*

$$\Phi_{\mathcal{K}}^{HKR} = \Phi^{H}_{\mathrm{td}(\mathcal{Z}/\mathcal{Y})\,\mathrm{ch}(\mathcal{K})}.$$

Proof. Let $(f, g) : \mathcal{Z} \to X \times \mathcal{Y}$ be a morphism. By [Kre09, Proposition 5.1], one can find a finite flat surjective morphism $\pi : Y \to \mathcal{Y}$ such that Y is a smooth projective scheme. From which we get a fiber product

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "\pi|_Z"'] & Y \arrow[d, "\pi"] \\ \mathcal{Z} \arrow[r, "g"] & \mathcal{Y}. \end{tikzcd}$$

We have the following chain of equalities,

$$\begin{aligned}
\Phi_{\mathcal{K}}^{\mathrm{HKR}} &= \frac{1}{\deg \pi}\pi_* \circ \phi_{\mathrm{HKR}} \circ \pi^* \circ (\Phi_{\mathcal{K}})_* \circ \phi_{\mathrm{HKR}}^{-1} \\
&= \frac{1}{\deg \pi}\pi_* \circ \phi_{\mathrm{HKR}} \circ (\Phi_{\mathbb{L}\pi|_Z^*\mathcal{K}})_* \circ \phi_{\mathrm{HKR}}^{-1} \\
&= \frac{1}{\deg \pi}\pi_* \circ \phi_{\mathrm{HKR}} \circ (\Phi_{\mathbb{R}(f,h)_*\mathbb{L}\pi|_Z^*\mathcal{K}})_* \circ \phi_{\mathrm{HKR}}^{-1} \\
&= \frac{1}{\deg \pi}\pi_* \circ \Phi^{\mathrm{H}}_{\mathrm{td}(X\times Y/Y)\,\mathrm{ch}(\mathbb{R}(f,h)_*\mathbb{L}\pi|_Z^*\mathcal{K})} \\
&= \frac{1}{\deg \pi}\pi_*\pi^* \circ \Phi^{\mathrm{H}}_{\mathrm{td}(X)\,\mathrm{ch}(\mathbb{R}(f,g)_*\mathcal{K})} \\
&= \Phi^{\mathrm{H}}_{\mathrm{td}(X\times\mathcal{Y}/\mathcal{Y})\,\mathrm{ch}(\mathbb{R}(f,g)_*\mathcal{K})} \\
&= \Phi^{\mathrm{H}}_{\mathrm{td}(\mathcal{Z}/\mathcal{Y})\,\mathrm{ch}\,\mathcal{K}}.
\end{aligned}$$

The first line is by definition. The second line follows from flat base change. The third line follows from the projection formula. The main justification is the fourth line which is a combination of Theorem 2 and Theorem 6 of [Ram10]. The fifth line is flat base change. The sixth line is [Ful13, Corollary 18.1.2]. The seventh line is [Edi12, Theorem 3.5]. □

2.5.2. *Factorization categories and the derived category.* Let X be a smooth variety and G be an affine algebraic group. Let T denote the total space of a G-equivariant vector bundle $\mathcal{E}$ on X, and let f be a regular equivariant section of $\mathcal{E}^\vee$ on X. Let $Z := Z(f)$ be the zero locus of f. Fiberwise dilation gives a $\mathbb{C}^*$-action on T. Furthermore, T inherits a function w which corresponds to the pairing with f. We have the following maps:

$$\begin{tikzcd} T|_Z \arrow[r, "i"] \arrow[d, bend right, "\pi|_Z"'] & T \\ Z \arrow[u, bend right, "j'"'] \arrow[ur, "j"] \arrow[r, "f"] & X \end{tikzcd} \tag{2.2}$$

Theorem 2.5.2 (Isik, Shipman, Hirano)**.** *There is an equivalence of categories*

$$\phi_+ := \mathbb{R}i_*\mathbb{L}\pi|_Z^*\colon\ \mathrm{D}([Z/G]) \to \mathrm{D}([T/G\times\mathbb{C}^*], w).$$

with inverse

$$\phi_+^{-1} = \mathbb{R}(\pi|_Z)_*\mathbb{L}i^* \overset{\mathbb{L}}{\otimes} \det\mathcal{E}^\vee[-\operatorname{rank}\mathcal{E}]\colon\ \mathrm{D}([T/G\times\mathbb{C}^*], w) \to \mathrm{D}([Z/G]).$$

Furthermore, if $M \subseteq [Z/G]$ is a closed proper substack, then ϕ_+, ϕ_+^{-1} restrict to an equivalence

$$\mathrm{D}([Z/G])_M \cong \mathrm{D}([T/G\times\mathbb{C}^*], w)_{[M/\mathbb{C}^*]}$$

where $[M/\mathbb{C}^*]$ is viewed as a substack of $[T/G\times\mathbb{C}^*]$ by inclusion along the zero section.

Proof. The equivalence is the main theorem of Isik's paper [Isi12] where the functor was defined differently. It was proven independently by Shipman [Shi12] who defined the functor this way. The generalization we are using is due to Hirano, see [Hir17a, Proposition 4.8]. The fact that the inverse functor is so defined, is because this functor is the adjoint, see e.g. [EP15, Theorem 3.8], [Pol16, Corollary 2.5.7], or [Hir17b, Theorem 4.36].

For the final statement, notice that if A is supported on M then $\mathbb{R}i_*\mathbb{L}\pi|_Z^*A$ is supported on $[T|_M/G\times\mathbb{C}^*]$. On the other hand, every factorization is supported on the critical locus of w which is contained in the zero section of $[T/G\times\mathbb{C}^*]$. Hence, $\mathbb{R}i_*\mathbb{L}\pi|_Z^*A$ is supported on $[M/\mathbb{C}^*]$. Conversely, if B is supported on $[M/\mathbb{C}^*]$ then $\mathbb{R}(\pi|_Z)_*\mathbb{L}i^*B\overset{\mathbb{L}}{\otimes}\det\mathcal{E}^\vee[-\operatorname{rank}\mathcal{E}]$ is supported on M. ◻

We will use $\phi_- := \mathbb{R}i_*\mathbb{L}\pi|_Z^*\colon \mathrm{D}([Z/G]) \to \mathrm{D}([T/G\times\mathbb{C}^*],-w)$ to denote the analogous functor but where we now view $\mathbb{R}i_*$ as mapping to $\mathrm{D}([T/G\times\mathbb{C}^*],-w)$.

Lemma 2.5.3. *Let $M\subseteq[Z/G]$ be a closed proper substack. The following diagram is commutative*

$$\begin{tikzcd}
\mathrm{D}([T/G\times\mathbb{C}^*],w)_{[M/\mathbb{C}^*]} \arrow[r, "(-\overset{\mathbb{L}}{\otimes}\phi_-(P))"] & \mathrm{D}([T/G\times\mathbb{C}^*],0)_{[M/\mathbb{C}^*]}. \arrow[dr, "\mathbb{R}\pi_*"] & \\
\mathrm{D}([Z/G])_M \arrow[u, "\phi_+"] \arrow[r, "(-\overset{\mathbb{L}}{\otimes}P\otimes\det\mathcal{E})[\operatorname{rank}\mathcal{E}]"'] & \mathrm{D}([Z/G])_M \arrow[r, "\mathbb{R}f_*"] & \mathrm{D}([X/G])_M.
\end{tikzcd} \tag{2.3}$$

Proof. We have the following functorial sequence of isomorphisms

$$\begin{aligned}
\mathbb{R}\pi_*(\phi_+(Q)\overset{\mathbb{L}}{\otimes}\phi_-(P)) &= \mathbb{R}\pi_*(\mathbb{R}i_*\pi|_Z^*Q\overset{\mathbb{L}}{\otimes}\mathbb{R}i_*\pi_{|}Z^*P)\\
&= \mathbb{R}\pi_*\mathbb{R}i_*(\pi|_Z^*Q\overset{\mathbb{L}}{\otimes}\mathbb{L}i^*\mathbb{R}i_*\pi|_Z^*P)\\
&= \mathbb{R}f_*\mathbb{R}(\pi|_Z)_*(\pi|_Z^*Q\overset{\mathbb{L}}{\otimes}\mathbb{L}i^*\mathbb{R}i_*\pi|_Z^*P)\\
&= \mathbb{R}f_*(Q\overset{\mathbb{L}}{\otimes}\mathbb{R}(\pi|_Z)_*\mathbb{L}i^*\mathbb{R}i_*\pi|_Z^*P)\\
&= \mathbb{R}f_*(Q\overset{\mathbb{L}}{\otimes}\phi^{-1}(\phi(P))\overset{\mathbb{L}}{\otimes}\det\mathcal{E})[\operatorname{rank}\mathcal{E}]\\
&= \mathbb{R}f_*(Q\overset{\mathbb{L}}{\otimes}P\overset{\mathbb{L}}{\otimes}\det\mathcal{E})[\operatorname{rank}\mathcal{E}]
\end{aligned}$$

where we use Proposition 2.2.10 in lines 2 and 4. ◻

Definition 2.5.4. *Define the equivalence $\tilde\phi_{+/-}\colon \mathrm{D}([Z/G])\to\mathrm{D}([T/G\times\mathbb{C}^*_R],w)$ to be*

$$\tilde\phi_{+/-} := \det(\mathcal{E}^\vee)\overset{\mathbb{L}}{\otimes}\phi_{+/-}[-\operatorname{rank}\mathcal{E}].$$

Remark 2.5.5. The previous proposition can be restated as

$$\mathbb{R}\pi_*(\tilde\phi_+(P)\overset{\mathbb{L}}{\otimes}\phi_-(Q)) = \mathbb{R}f_*(P\overset{\mathbb{L}}{\otimes}Q).$$

On $[T/G]$, the pull-back vector bundle $\pi^*\mathcal{E}$ is naturally $\mathbb{C}^*_R$-equivariant and has the $\mathbb{C}^*_R$-invariant tautological section $\mathrm{taut} \in H^0([T/G], \pi^*\mathcal{E})$. We also have the section $\pi^* f \in H^0([T/G], \pi^*(\mathcal{E}^\vee \otimes_{\mathcal{O}_{[T/G]}} \mathcal{O}_{[T/G]}(\eta)))$ where η is the character of $\mathbb{C}^*_R$ of weight one. The composition

$$\mathcal{O}_{[T/G]} \xrightarrow{\mathrm{taut}} \pi^*\mathcal{E} \xrightarrow{\pi^*(f)^\vee \otimes \mathrm{id}_{\mathcal{O}_{[T/G]}(\eta)}} \mathcal{O}_{[T/G]}(\eta)$$

is equal to w, hence we get a Koszul factorization

$$S_1 := \{\pi^*(f)^\vee, \mathrm{taut}\} \in \mathrm{D}([T/G \times \mathbb{C}^*_R], w). \tag{2.4}$$

Remark 2.5.6. We note for future reference that $\phi_+(\mathcal{O}_Z) = S_1^\vee = \{\mathrm{taut}^\vee, \pi^*(f)\}$ and consequently,

$$\tilde{\phi}_+(\mathcal{O}_Z) = \det(\mathcal{E}^\vee) \overset{\mathbb{L}}{\otimes} S_1^\vee[-\operatorname{rank}\mathcal{E}] = S_1.$$

We introduce a Todd class correction to get the correct identification of Hochschild homology $\mathrm{HH}_*([T/G \times \mathbb{C}^*_R], w)$ with the cohomology of Z.

Definition 2.5.7. *Define the isomorphism*

$$\varphi^{\mathrm{td}}_* : \mathrm{H}^*(Z) \to \mathrm{HH}_*([T/G \times \mathbb{C}^*_R], w)$$

by

$$\varphi^{\mathrm{td}}_*(\gamma) := (\tilde{\phi}_+)_* \circ \phi^{-1}_{\mathrm{HKR}}(\mathrm{td}(\mathcal{E}) \cup \gamma).$$

2.5.3. *Comparing localized Chern characters.* Consider a vector bundle A on a quotient Deligne–Mumford stack S. Equip S with a trivial $\mathbb{C}^*$-action and suppose we have a $\mathbb{C}^*$-action on $\operatorname{tot} A$ which is equivariant with respect to the projection $p\colon \operatorname{tot} A \to S$. This yields a fiber square,

$$\begin{tikzcd} \operatorname{tot} A \arrow[r, "\pi_A"] \arrow[d, "p"] & {[\operatorname{tot} A/\mathbb{C}^*]} \arrow[d, "p"] \\ S \arrow[r, "\pi_S"] & {[S/\mathbb{C}^*].} \end{tikzcd} \tag{2.5}$$

Let X be a closed substack of S. Using the equivalence,

$$\mathrm{D}([S/\mathbb{C}^*], 0) = \mathrm{D}(S)$$

we can view $\mathbb{R}p_*$ as a functor

$$\mathbb{R}p_*\colon\ \mathrm{D}([\operatorname{tot} A/\mathbb{C}^*], 0)_X \to \mathrm{D}(S)_X.$$

Proposition 2.5.8. *Suppose that S is smooth over Spec$(\mathbb{C})$. There is a commutative diagram*

$$\begin{tikzcd} \mathrm{D}([\operatorname{tot} A/\mathbb{C}^*], 0)_X \arrow[r, "\mathbb{L}\pi_A^*"] \arrow[d, "\mathbb{R}p_*"] & \mathrm{D}(\operatorname{tot} A, 0)_X \arrow[r, "{}^{\mathbb{Z}_2}\mathrm{ch}_X^{\operatorname{tot} A}"] & A^*(X \to \operatorname{tot} A)_{\mathbb{Q}} \arrow[d, "{[p]\mathrm{td}(A)}"] \\ \mathrm{D}(S)_X \arrow[rr, "\mathrm{ch}_X^S"] & & A^*(X \to S)_{\mathbb{Q}} \end{tikzcd}$$

where $[p]$ is the canonical orientation of the flat map p.

Proof. Let ${}^{\mathbb{Z}_2}K_0(\operatorname{tot} A)_X$ (resp. ${}^{\mathbb{Z}_2}K_0(S)_X$) denote the Grothendieck group of the abelian category of two periodic cochain complexes of coherent sheaves on $\operatorname{tot} A$ (resp. S) which are exact on $\operatorname{tot} A \setminus X$ (resp. $S \setminus X$). We can view the result as coming from the following extended diagram.

$$\begin{tikzcd}
\mathrm{D}([\operatorname{tot} A/\mathbb{C}^*], 0)_X \arrow[r, "\mathbb{L}\pi_A^*"] \arrow[d, "\mathbb{R}p_*"] & \mathrm{D}(\operatorname{tot} A, 0)_X \arrow[r] \arrow[d, "\mathbb{R}p_*"] & {}^{\mathbb{Z}_2}K_0(\operatorname{tot}(A))_X \arrow[r, "{}^{\mathbb{Z}_2}\mathrm{ch}_X^{\operatorname{tot} A}"] \arrow[d, "\mathbb{R}p_*"] & A^*(X \to \operatorname{tot} A)_{\mathbb{Q}} \arrow[d, "{[p]\mathrm{td}(A)}"] \\
\mathrm{D}(S)_X \arrow[r, "\mathbb{L}\pi_S^*"] & \mathrm{D}(S, 0)_X \arrow[r] & {}^{\mathbb{Z}_2}K_0(S)_X \arrow[r, "{}^{\mathbb{Z}_2}\,\mathrm{ch}_X^S"] & A^*(X \to S)_{\mathbb{Q}}
\end{tikzcd}$$

The left commutative square comes from flat base change from (2.5) and the middle square is obvious. Now the horizontal arrow along the bottom is just ch_X^S by [PV01, Proposition 2.2]. The commutativity on the right square is a localized version of [Chi06, Lemma 5.3.8] which we now prove.

Let

$$W_\bullet := \ldots \xrightarrow{d_0} W_1 \xrightarrow{d_1} W_0 \xrightarrow{d_0} W_1 \xrightarrow{d_1} \ldots$$

be a 2-periodic complex of coherent sheaves on $\operatorname{tot} A$ supported on X. Consider the morphism of sheaves on $\operatorname{tot} A \times \mathbb{A}^1$

$$\begin{aligned} H : (W_1 \oplus \operatorname{im} d_1)[t] &\to \operatorname{im} d_1[t] \\ (w, v) &\mapsto d_1(w) - tv \\ t &\mapsto t \end{aligned}$$

We have the following diagram

$$\begin{tikzcd}
 & & W_0[t] \arrow[dll, dashed, "f"'] \arrow[d, "g"] \arrow[dr, "0"] & & \\
0 \arrow[r] & \ker H \arrow[r] & (W_1 \oplus \operatorname{im} d_1)[t] \arrow[r] & \operatorname{im} d_1[t] \arrow[r] & 0
\end{tikzcd}$$

where $g = (d_0, 0)$ extended by t. We also have a map

$$\begin{aligned} s : \ker H &\to W_0[t] \\ (w, v) &\mapsto d_1(w) \\ t &\mapsto t. \end{aligned}$$

This gives a 2-periodic complex of coherent sheaves on $\operatorname{tot} A \times \mathbb{A}^1$ supported on $X \times \mathbb{A}^1$,

$$\widetilde{W}_\bullet := \ldots \xrightarrow{f} \ker H \xrightarrow{s} W_0[t] \xrightarrow{f} \ker H \xrightarrow{s} \ldots$$

By Lemma 2.1 and Lemma 2.3 (iii) of [PV16],

$${}^{\mathbb{Z}_2}\mathrm{ch}_X^{\operatorname{tot} A}(\widetilde{W}_\bullet|_0) = {}^{\mathbb{Z}_2}\mathrm{ch}_X^{\operatorname{tot} A}(\widetilde{W}_\bullet|_1).$$

Now notice that

$$\widetilde{W}_\bullet|_1 = W_\bullet$$

and

$$\widetilde{W}_\bullet|_0 = \ldots \to W_1 \xrightarrow{s} \ker d_1 \oplus \operatorname{im} d_1 \to W_1 \to \ldots$$

Arguing similarly to replace W_1, we get

$$^{\mathbb{Z}_2}\mathrm{ch}_X^{\mathrm{tot}\,A}(W_\bullet) = {}^{\mathbb{Z}_2}\mathrm{ch}_X^{\mathrm{tot}\,A}(W'_\bullet)$$

where

$$\widetilde{W'_\bullet} = \ldots \to \ker d_0 \oplus \operatorname{im} d_0 \xrightarrow{s} \ker d_1 \oplus \operatorname{im} d_1 \to \ker d_0 \oplus \operatorname{im} d_0 \to \ldots$$

This complex is the 2-periodic folding of the complex

$$C_\bullet := 0 \to \operatorname{im} d_0 \to \ker d_0 \oplus \operatorname{im} d_1 \to \ker d_1 \to 0.$$

Hence, by Proposition 2.2 of [PV16],

$$^{\mathbb{Z}_2}\mathrm{ch}_X^{\mathrm{tot}\,A}(W'_\bullet) = \mathrm{ch}_X^{\mathrm{tot}\,A}(C_\bullet)$$

Finally $C_\bullet$ is quasi-isomorphic to

$$0 \to H^0(W_\bullet) \xrightarrow{0} H^1(W_\bullet) \to 0.$$

Let $i : S \to \mathrm{tot}\,A$ be the inclusion along the zero section. We claim that, $\mathbb{R}i_*\mathbb{R}p_*$ is the identity map on $\mathrm{K}_0(\mathrm{tot}\,A)_X$. This is well-known but we outline the proof. Namely, note that $\mathbb{R}p_*\mathbb{R}i_* = \mathrm{Id}$, hence $\mathbb{R}i_*$ is injective and it is enough to prove it is also surjective. This follows from the fact that any coherent sheaf on $\mathrm{tot}\,A$ set-theoretically supported on X has a filtration whose associated graded pieces are scheme-theoretically supported on X. The claim follows.

Hence,

$$\begin{aligned}\mathrm{ch}_X^{\mathrm{tot}\,A}(C_\bullet) &= \mathrm{ch}_X^{\mathrm{tot}\,A}(H^0(W_\bullet)) - \mathrm{ch}_X^{\mathrm{tot}\,A}(H^1(W_\bullet)) \\ &= \mathrm{ch}_X^{\mathrm{tot}\,A}(\mathbb{R}i_*\mathbb{R}p_*H^0(W_\bullet)) - \mathrm{ch}_X^{\mathrm{tot}\,A}(\mathbb{R}i_*\mathbb{R}p_*H^1(W_\bullet)) \\ &= (\mathrm{ch}_X^S(\mathbb{R}p_*H^0(W_\bullet)) - \mathrm{ch}_X^S(\mathbb{R}p_*H^1(W_\bullet)))\,\mathrm{td}(A)^{-1}[p].\end{aligned}$$

The first line is by the claim above. The second line is Corollary 18.1.2 of [Ful13]. This completes the proof. $\square$

3. Admissible resolutions of GLSMs

In this section, we construct a distinguished factorization, that we call a PV factorization due to its initial use by Polishchuk–Vaintrob in [PV16].

3.1. **Setup.** We start with a (partial) GLSM data $(V, G, \mathbb{C}^*_R, w)$, *without* the choice of a linearization θ. Recall from §1 that we have the groups

$$\Gamma := G \cdot \mathbb{C}^*_R \subseteq GL(V)$$

and a character χ entering in the exact sequence

$$1 \to G \to \Gamma \xrightarrow{\chi} \mathbb{C}^* \to 1.$$

Note that we may consider V either as an algebraic object, i.e., as a Γ-representation, or as a geometric object, i.e., as the affine space $\mathrm{Spec}\,\mathrm{Sym}\,V^\vee$ with the linear action of Γ. We will switch between these two viewpoints without further comment when convenient.

3.1.1. *Some moduli stacks.*

Definition 3.1.1. *Let $\mathfrak{M}^{\mathrm{tw}}_{g,r}$ be the moduli stack parametrizing families of (balanced) prestable twisted curves (in the sense of* [AV02]*) with r gerbe markings, see* [Ols07].

By [Ols07, Theorems 1.9, 1.10], $\mathfrak{M}^{\mathrm{tw}}_{g,r}$ is a smooth Artin stack, locally of finite type. The fibered product of the universal gerbes over $\mathfrak{M}^{\mathrm{tw}}_{g,r}$ gives the smooth Artin stack

$$\mathfrak{M}^{\mathrm{orb}}_{g,r} := \mathscr{G}_1 \times_{\mathfrak{M}^{\mathrm{tw}}_{g,r}} \mathscr{G}_2 \times_{\mathfrak{M}^{\mathrm{tw}}_{g,r}} \cdots \times_{\mathfrak{M}^{\mathrm{tw}}_{g,r}} \mathscr{G}_r, \tag{3.1}$$

which parametrizes families of marked twisted curves together with sections of the gerbe markings. These are precisely the "families of prestable orbi-curves" appearing in Definition 1.2.1.

Remark 3.1.2. To avoid confusion, in what follows we will use the terms "twisted curves" to refer to families parametrized by Definition 3.1.1 and "orbi-curves" to refer to families parametrized by (3.1).

Definition 3.1.3. *For any reductive complex algebraic group G, we let $\mathfrak{M}^{\mathrm{orb/tw}}_{g,r}(BG)$ denote the stack parametrizing families of prestable orbi-curves (resp. twisted curves) $\mathcal{C} \to T$, with T a scheme, together with a principal G-bundle on $\mathcal{C}$, such that the induced morphism $\mathcal{C} \to BG$ is representable.*

Definition 3.1.4. *Define the stack $\mathfrak{M}^{\mathrm{orb/tw}}_{g,r}(B\Gamma)_{\omega^{\log}_{\mathfrak{C}}}$ parametrizing families of prestable orbi-curves (resp. twisted curves) together with a principal Γ-bundle P and an isomorphism $\chi_*(P) \cong \mathring{\omega}^{\log}_{\mathfrak{C}}$, such that the induced map $[P] : \mathcal{C} \to B\Gamma$ is representable.*

These are again smooth Artin stacks, locally of finite type (for proofs, see, e.g., [CKM14, Proposition 2.1.1], [CCK15, §2], and [FJR17, Lemma 5.2.2]).

There are obvious forgetful smooth morphisms

$$\mathfrak{M}^{\mathrm{orb/tw}}_{g,r}(BG) \to \mathfrak{M}^{\mathrm{orb/tw}}_{g,r}, \qquad \mathfrak{M}^{\mathrm{orb/tw}}_{g,r}(B\Gamma)_{\omega^{\log}_{\mathfrak{C}}} \to \mathfrak{M}^{\mathrm{orb/tw}}_{g,r}(B\Gamma) \to \mathfrak{M}^{\mathrm{orb/tw}}_{g,r},$$

under which the universal curves $\mathfrak{C}$ (together with the gerbes with sections) pull back.

3.1.2. *Stacks mapping to $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma)_{\omega^{\log}_{\mathfrak{C}}}$.* In this section we will work with an Artin stack S with a map $S \to \mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma)_{\omega^{\log}_{\mathfrak{C}}}$ by which we pull-back the universal structures on $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma)_{\omega^{\log}_{\mathfrak{C}}}$. To fix notation, we describe these explicitly. First, we have a flat and proper family

$$\pi \colon \mathfrak{C} \to S$$

of r-pointed prestable twisted curves over S, together with sections $S \to \mathscr{G}_i$ of the r disjoint gerbe markings $\mathscr{G}_i \subseteq \mathfrak{C}$. We denote $\Sigma_i : \mathscr{G}_i \hookrightarrow \mathfrak{C}$ the inclusion map. For any T-point $T \to S$, with T a scheme, we have a family of r-pointed prestable orbi-curves over T by pull-back. We require that each section $T \to \mathscr{G}_{iT}$ induces an isomorphism of T with the coarse moduli space of $\mathscr{G}_i$. It follows that the composition $\pi_T \circ \Sigma_{iT} : \mathscr{G}_{iT} \to T$ is the coarse moduli map. In particular, we have a canonical isomorphism,

$$\mathbb{R}(\pi \circ \Sigma_i)_* \mathcal{O}_{\mathscr{G}_i} \cong (\pi \circ \Sigma_i)_* \mathcal{O}_{\mathscr{G}_i} \cong \mathcal{O}_S$$

(where the first equality uses the fact that G is linearly reductive).

Denote $\mathscr{G} := \coprod_i \mathscr{G}_i \subseteq \mathfrak{C}$ the union of the gerbe markings, with inclusion map Σ. Each of $\mathscr{G}_i$ is an effective Cartier divisor in $\mathfrak{C}$, and $\mathscr{G} = \mathscr{G}_1 + ... + \mathscr{G}_r$ as Cartier divisors. Hence there are exact sequences

$$0 \to \mathcal{O}_{\mathfrak{C}}(-\mathscr{G}_i) \to \mathcal{O}_{\mathfrak{C}} \to \Sigma_{i*}\mathcal{O}_{\mathscr{G}_i} \to 0,$$
$$0 \to \mathcal{O}_{\mathfrak{C}}(-\mathscr{G}) \to \mathcal{O}_{\mathfrak{C}} \to \Sigma_{*}\mathcal{O}_{\mathscr{G}} \to 0.$$

For a quasicoherent sheaf $\mathcal{F}$ on $\mathfrak{C}$ we denote

$$\mathcal{F}|_{\mathscr{G}_i} := \Sigma_{i*}\Sigma_i^*\mathcal{F}, \quad \mathcal{F}|_{\mathscr{G}} := \Sigma_*\Sigma^*\mathcal{F}.$$

If $\mathcal{F}$ is locally free we have induced short exact sequences,

$$0 \to \mathcal{F}(-\mathscr{G}) \to \mathcal{F} \to \mathcal{F}|_{\mathscr{G}} \to 0.$$

Since $\mathscr{G}_i = S \times B\mu_{m_i}$ for some positive integers m_i, $\Sigma_i^*\mathcal{F}$ can be considered as a vector bundle on S with fiberwise linear action by μ_{m_i}. Hence, for each i, $\pi_*(\mathcal{F}|_{\mathscr{G}_i})$ is the μ_{m_i}-invariant part of the vector bundle.

Recall that for the sheaf of relative log differentials, $\omega_{\mathfrak{C}}^{\log} := \omega_{\mathfrak{C}}(\mathscr{G}_1 + ... + \mathscr{G}_r)$, there are canonical isomorphisms

$$\Sigma_i^*\omega_{\mathfrak{C}}^{\log} \cong \mathcal{O}_{\mathscr{G}_i}, \tag{3.2}$$

and therefore induced canonical isomorphisms

$$\pi_*(\omega_{\mathfrak{C}}^{\log}|_{\mathscr{G}_i}) \cong \mathcal{O}_S, \quad \pi_*(\omega_{\mathfrak{C}}^{\log}|_{\mathscr{G}}) \cong \mathcal{O}_S^r.$$

Further, we are given a principal Γ-bundle P on $\mathfrak{C}$, *and* a fixed isomorphism of principal $\mathbb{C}^*$-bundles,

$$\varkappa\colon P \times_\Gamma \mathbb{C}^* \to \mathring{\omega}_{\mathfrak{C}}^{\log},$$

where Γ acts on $\mathbb{C}^*$ by the character χ. We abuse notation and will use the same letter for the induced isomorphism of line bundles, $\varkappa : \mathcal{L}_\chi \to \omega_{\mathfrak{C}}^{\log}$.

The data $(\mathfrak{C} \to S, \mathscr{G}, P, \varkappa)$ coming from the map $S \to \mathfrak{M}_{g,r}^{\mathrm{orb}}(B\Gamma)_{\omega_{\mathfrak{C}}^{\log}}$ couples with the Γ-representation V and the G-invariant potential w to give additional structures.

3.1.3. *Coupling with V.* Let $\mathcal{V}$ denote the locally-free sheaf of sections of the geometric vector bundle $P \times_\Gamma V$. Set, as in Definition 1.2.1,

$$\operatorname{tot}\mathcal{V} := \operatorname{Spec}(\operatorname{Sym}\mathcal{V}^\vee) = P \times_\Gamma V.$$

We claim that $\mathcal{V}$ comes with "evaluation maps" at the markings to the inertia stack $I[V/G]$. To see this, fix an index $i \in \{1, \dots, r\}$ and consider the total space

$$\operatorname{tot}(\pi_*\mathcal{V}|_{\mathscr{G}_i}) \xrightarrow{p_i} S,$$

with its family of curves with gerbe markings

$$\begin{tikzcd}
\widetilde{\mathscr{G}}_i \arrow[r, "r_i"] \arrow[d, "\widetilde{\Sigma}_i"'] & \mathscr{G}_i \arrow[d, "\Sigma_i"] \\
\widetilde{\mathfrak{C}} \arrow[r, "q_i"] \arrow[d, "\widetilde{\pi}"'] & \mathfrak{C} \arrow[d, "\pi"] \\
\operatorname{tot}(\pi_*\mathcal{V}|_{\mathscr{G}_i}) \arrow[r, "p_i"] & S
\end{tikzcd}$$

obtained by pull-back from S.

Denote

$$\pi_i := \pi \circ \Sigma_i : \mathscr{G}_i \to S, \quad \widetilde{\pi}_i := \widetilde{\pi} \circ \widetilde{\Sigma_i} : \widetilde{\mathscr{G}_i} \to \mathrm{tot}(\pi_* \mathcal{V}|_{\mathscr{G}_i}),$$

the projections from the gerbes to their respective bases, and set

$$\mathcal{V}_i := \Sigma_i^* \mathcal{V}, \quad \widetilde{\mathcal{V}}_i := \widetilde{\Sigma}_i^* q_i^* \mathcal{V} = r_i^* \mathcal{V}_i.$$

There is a base-change isomorphism

$$p_i^*(\pi_i)_* \mathcal{V}_i \cong (\widetilde{\pi}_i)_* r_i^* \mathcal{V}_i = (\widetilde{\pi}_i)_* \widetilde{\mathcal{V}}_i. \tag{3.3}$$

The vector bundle $p_i^*(\pi_i)_* \mathcal{V}_i = p_i^*(\pi_* \mathcal{V}|_{\mathscr{G}_i})$ on $\mathrm{tot}(\pi_* \mathcal{V}|_{\mathscr{G}_i})$ has the tautological section

$$\tau_i \in H^0(\mathrm{tot}(\pi_* \mathcal{V}|_{\mathscr{G}_i}), p_i^*(\pi_i)_* \mathcal{V}_i).$$

By (3.3) and the identification

$$H^0(\widetilde{\mathscr{G}}_i, \widetilde{\mathcal{V}}_i) = H^0(\mathrm{tot}(\pi_* \mathcal{V}|_{\mathscr{G}_i}), (\widetilde{\pi}_i)_* \widetilde{\mathcal{V}}_i),$$

we may view τ_i as a global section in $H^0(\widetilde{\mathscr{G}}_i, \widetilde{\mathcal{V}}_i)$.

Let

$$\widetilde{P}_i := \widetilde{\Sigma}_i^* q_i^* P = r_i^* \Sigma_i^* P$$

be the pulled-back principal Γ-bundle on $\widetilde{\mathscr{G}}_i$, so that $\widetilde{\mathcal{V}}_i$ is the sheaf of sections of the geometric vector bundle $\widetilde{P}_i \times_\Gamma V$. The pair $(\widetilde{P}_i, \tau_i)$ gives rise to a morphism

$$[\widetilde{P}_i, \tau_i] \colon \widetilde{\mathscr{G}}_i \to [V/\Gamma].$$

It is representable, since $[\widetilde{P}_i] \colon \widetilde{\mathscr{G}}_i \to B\Gamma$ is so. Further, being the pull-back of $\mathscr{G}_i$, the gerbe $\widetilde{\mathscr{G}}_i$ comes with a section. It follows that the diagram

$$\begin{tikzcd} \widetilde{\mathscr{G}}_i \arrow[r, "{[\widetilde{P}_i, \tau_i]}"] \arrow[d, "\widetilde{\pi}_i"'] & {[V/\Gamma]} \\ \mathrm{tot}(\pi_* \mathcal{V}|_{\mathscr{G}_i}) & \end{tikzcd}$$

determines a morphism

$$\mathrm{tot}(\pi_* \mathcal{V}|_{\mathscr{G}_i}) \to I[V/\Gamma], \tag{3.4}$$

see [AGV08, §3.2]. Here $I\mathfrak{X}$ denotes the inertia stack $\mathfrak{X} \times_{\mathfrak{X} \times \mathfrak{X}} \mathfrak{X}$ of an algebraic stack $\mathfrak{X}$. We allow that $I\mathfrak{X}$ is not quasi-compact.

We claim that the map (3.4) factors through the natural map $I[V/G] \to I[V/\Gamma]$. Indeed, since $\widetilde{P}_i \times_\Gamma \mathbb{C}^*$ is canonically isomorphic to $\widetilde{P}_i/G$, using the trivialization of $\widetilde{\Sigma}_i^* \omega_{\widetilde{\mathfrak{C}}}^{\log}$ coming from (3.2), we obtain a principal G-bundle $\widetilde{P'}_i$ on $\widetilde{\mathscr{G}}_i$ such that $\widetilde{P'}_i \times_G \Gamma \cong \widetilde{P}_i$ as Γ-bundles. It follows that there is an isomorphism of geometric vector bundles $\widetilde{P'}_i \times_G V \cong \widetilde{P}_i \times_\Gamma V$, and therefore $\widetilde{P'}_i \times_G V$ comes with the section τ_i. In other words, we have constructed a diagram

$$\begin{tikzcd} \widetilde{\mathscr{G}}_i \arrow[r, "{[\widetilde{P'}_i, \tau_i]}"] \arrow[d, "\widetilde{\pi}_i"'] & {[V/G]} \\ \mathrm{tot}(\pi_* \mathcal{V}|_{\mathscr{G}_i}) & \end{tikzcd}$$

which determines the required morphism

$$\mathrm{ev}_{\mathcal{V}}^i : \mathrm{tot}(\pi_* \mathcal{V}|_{\mathscr{G}_i}) \to I[V/G], \tag{3.5}$$

factoring (3.4).

3.1.4. *Coupling with w.* Equip $\mathbb{A}^1$ with the standard dilation action of $\mathbb{C}^*$ and view the potential as a function

$$w\colon V \to \mathbb{A}^1$$

which is Γ-equivariant with respect to the map $\chi : \Gamma \to \mathbb{C}^*$. This amounts to giving a Γ-invariant element in $w \in \mathbb{C}_\chi \otimes \operatorname{Sym} V^\vee$. Pairing with this w gives a morphism of Γ-representations $\overline{w} : \operatorname{Sym} V \to \mathbb{C}_\chi$. Then, viewing it as a morphism of vector bundles on the classifying stack $B\Gamma$ and pulling back via the map $[P] : \mathfrak{C} \to B\Gamma$ gives a homomorphism $[P]^*\overline{w} : \operatorname{Sym} \mathcal{V} \to \mathcal{L}_\chi$ of locally free sheaves on $\mathfrak{C}$.

Define

$$\varkappa_w\colon \operatorname{Sym} \mathcal{V} \to \omega_{\mathfrak{C}}^{\log} \tag{3.6}$$

as the composition

$$\operatorname{Sym} \mathcal{V} \xrightarrow{[P]^*\bar{w}} \mathcal{L}_\chi \xrightarrow{\varkappa} \omega_{\mathfrak{C}}^{\log}.$$

Remark 3.1.5. In the discussion of the evaluation maps and of $\varkappa_w$ we have essentially ignored the $\mathbb{C}_R^*$-action, i.e., the *lower* grading of V. This was done for the purpose of streamlining the exposition in subsections §3.2-§3.5 below. However, this action is crucial for the theory developed in the paper and we will take it into account fully in §3.7

3.2. **Admissible resolutions.** Admissible resolutions are special resolutions of $\mathbb{R}\pi_*\mathcal{V}$ on S satisfying three conditions, see Definition 3.2.1. We prove that the first condition is valid over any base Artin stack $S \to \mathfrak{M}_{g,r}(B\Gamma)_{\omega_{\mathfrak{C}}^{\log}}$ for which the map $\pi : \mathfrak{C} \to S$ is projective, see §3.4. However, to obtain the second and third conditions, we need to work over a Deligne–Mumford stack S whose coarse moduli space is projective over an affine, as we show in §3.5. In §3.3, we explain how to construct a fundamental factorization from an admissible resolution.

3.2.1. *Morphisms in the derived category.* We construct two morphisms in the bounded derived category $\mathrm{D}(S)$ of coherent sheaves on S.

(a) The first morphism exists when there is at least one marking and is easy to define. Namely, for each $i = 1, \dots, r$, the restriction map $\mathrm{res}_i : \mathcal{V} \to \mathcal{V}|_{\mathscr{G}_i}$ induces a morphism

$$\begin{aligned}\mathbb{R}\pi_*\mathcal{V} \xrightarrow{\mathbb{R}\pi_*\,\mathrm{res}_i} &\mathbb{R}\pi_*(\mathcal{V}|_{\mathscr{G}_i})\\ &= \pi_*(\mathcal{V}|_{\mathscr{G}_i}).\end{aligned} \tag{3.7}$$

Hence we also have a morphism $\mathbb{R}\pi_*\,\mathrm{res} = \sum_{i=1}^r \mathbb{R}\pi_*\,\mathrm{res}_i$ in $\mathrm{D}(S)$

$$\mathbb{R}\pi_*\mathcal{V} \xrightarrow{\mathbb{R}\pi_*\,\mathrm{res}} \pi_*(\mathcal{V}|_{\mathscr{G}}) \cong \oplus_{i=1}^r \pi_*(\mathcal{V}|_{\mathscr{G}_i}). \tag{3.8}$$

(b) The other morphism is more involved. Let

$$\mathrm{natural}\colon \operatorname{Sym}\mathbb{R}\pi_*\mathcal{V} \to \mathbb{R}\pi_*\operatorname{Sym}\mathcal{V} \tag{3.9}$$

be the map coming from the counit of the adjunction $\mathbb{L}\pi^* \dashv \mathbb{R}\pi_*$. That is, natural is induced from the composition,

$$\mathbb{L}\pi^*\operatorname{Sym}\mathbb{R}\pi_*\mathcal{V} \to \operatorname{Sym}\mathbb{L}\pi^*\mathbb{R}\pi_*\mathcal{V} \to \operatorname{Sym}\mathcal{V}.$$

Composing with the derived push-forward of the map $\varkappa_w$, see (3.6), gives a morphism in the derived category

$$\begin{aligned}\operatorname{Sym}\mathbb{R}\pi_*\mathcal{V} &\xrightarrow{\text{natural}} \mathbb{R}\pi_*\operatorname{Sym}\mathcal{V}\\ &\xrightarrow{\mathbb{R}\pi_*\varkappa_w} \mathbb{R}\pi_*\omega_{\mathfrak{C}}^{\log}.\end{aligned} \tag{3.10}$$

We may combine (3.10) with the restriction map to $\mathbb{R}\pi_*\omega_{\mathfrak{C}}^{\log}|_{\mathscr{G}}$ (in the case of no markings this becomes the zero map). Since $\omega_{\mathfrak{C}}^{\log}$ trivializes on $\mathscr{G}$, we have $\mathbb{R}\pi_*\omega_{\mathfrak{C}}^{\log}|_{\mathscr{G}} \cong \mathcal{O}_S^r$. This gives a map

$$\operatorname{Sym}\mathbb{R}\pi_*\mathcal{V} \to \mathcal{O}_S^r\,. \tag{3.11}$$

The map above leads to the following morphism of exact triangles

$$\begin{tikzcd} E \arrow[r] \arrow[d, dashed] & \operatorname{Sym}\mathbb{R}\pi_*\mathcal{V} \arrow[r] \arrow[d] & \mathcal{O}_S^r \arrow[r] \arrow[d] & E[1] \arrow[d, dashed] \\ \mathbb{R}\pi_*\omega_{\mathfrak{C}} \arrow[r] & \mathbb{R}\pi_*\omega_{\mathfrak{C}}^{\log} \arrow[r] & \mathcal{O}_S^r \arrow[r] & \mathbb{R}\pi_*\omega_{\mathfrak{C}}[1] \end{tikzcd} \tag{3.12}$$

where

$$E := \operatorname{Cone}(\operatorname{Sym}\mathbb{R}\pi_*\mathcal{V} \to \mathcal{O}_S^r)[-1],$$

is the shifted cone. The dashed arrow is not uniquely determined in this fashion. Therefore, we need to provide a mechanism to determine it.

Let

$$E_d := \operatorname{Cone}(\operatorname{Sym}^d\mathbb{R}\pi_*\mathcal{V} \to \mathcal{O}_S^r)[-1],$$

and

$$\operatorname{Sym}^d\mathfrak{C} := [\overbrace{\mathfrak{C}\times_S \dots \times_S \mathfrak{C}}^{d-times}/S_d]$$

where S_d is the symmetric group on d letters acting in the obvious way. We have a S_d-equivariant diagonal map

$$\Delta_d : [\mathfrak{C}/S_d] \to \operatorname{Sym}^d\mathfrak{C}$$

which induces a map

$$\sigma_d : [\mathscr{G}/S_d] \to \operatorname{Sym}^d\mathfrak{C}.$$

This induces a morphism of exact sequences on $\operatorname{Sym}^d\mathfrak{C}$

$$\begin{tikzcd} 0 \arrow[r] & \ker \arrow[r] \arrow[d, "f_d"] & \mathcal{V}^{\boxtimes d} \arrow[r] \arrow[d] & (\sigma_d)_*\omega_{\mathfrak{C}}^{\log}|_{\mathscr{G}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (\sigma_d)_*\omega_{\mathfrak{C}} \arrow[r] & (\sigma_d)_*\omega_{\mathfrak{C}}^{\log} \arrow[r] & (\sigma_d)_*\omega_{\mathfrak{C}}^{\log}|_{\mathscr{G}} \arrow[r] & 0 \end{tikzcd} \tag{3.13}$$

Pushing forward and composing with the Grothendieck trace map we obtain our desired second map

$$\begin{aligned} E_d &\xrightarrow{\mathbb{R}\pi_* f_d} \mathbb{R}\pi_*\omega_{\mathfrak{C}} \\ &\xrightarrow{\mathrm{H}^1} \mathbb{R}^1\pi_*\omega_{\mathfrak{C}}[-1] \\ &\xrightarrow{\text{trace}} \mathcal{O}_S[-1] \end{aligned} \tag{3.14}$$

In the presence of markings, the morphism (3.8) and (3.11) satisfy a certain compatibility. Namely, apply the functor Sym to (3.8) and then compose with the inclusion

$$\operatorname{Sym}\pi_*(\mathcal{V}|_{\mathscr{G}_i}) \xrightarrow{\text{natural}} \pi_*(\operatorname{Sym}\mathcal{V}|_{\mathscr{G}_i})$$

to get

$$\operatorname{Sym}\mathbb{R}\pi_*\mathcal{V} \xrightarrow{\oplus_{i=1}^r \text{natural}\circ\operatorname{Sym}\mathbb{R}\pi_*\operatorname{res}_i} \oplus_{i=1}^r\pi_*(\operatorname{Sym}\mathcal{V}|_{\mathscr{G}_i}). \tag{3.15}$$

Consider the commutative diagram of quasicoherent sheaves on $\mathfrak{C}$

$$\begin{tikzcd} \operatorname{Sym}\mathcal{V} \arrow[r, "{[P]^*\bar{w}}"] \arrow[d, "\text{restrict}"'] & \mathcal{L}_\chi \arrow[r, "\varkappa"] \arrow[d, "\text{restrict}"'] & \omega_{\mathfrak{C}}^{\log} \arrow[d, "\text{restrict}"'] \\ \Sigma_*\Sigma^*\operatorname{Sym}\mathcal{V} \arrow[r, "{\Sigma_*\Sigma^*[P]^*\bar{w}}"] & \Sigma_*\Sigma^*\mathcal{L}_\chi \arrow[r, "\Sigma_*\Sigma^*\varkappa"] & \Sigma_*\Sigma^*\omega_{\mathfrak{C}}^{\log} \end{tikzcd}$$

Applying $\mathbb{R}\pi_*$ gives a commuting diagram in $\mathrm{D}(S)$

$$\begin{tikzcd} \mathbb{R}\pi_*\operatorname{Sym}\mathcal{V} \arrow[r, "\pi_*\varkappa_w"] \arrow[d] & \mathbb{R}\pi_*\omega_{\mathfrak{C}}^{\log} \arrow[d] \\ \oplus_{i=1}^r\pi_*((\operatorname{Sym}\mathcal{V})|_{\mathscr{G}_i}) \arrow[r, "\bigoplus_i \pi_*\Sigma_{i*}{\Sigma_i}^*\varkappa_w"] & \mathcal{O}_S^r \end{tikzcd}$$

where we have now omitted the middle column.

The natural map (3.9) followed by the clockwise composition from the top left to the bottom right gives the morphism (3.11). On the other hand, the natural map (3.9) followed by composition in the counterclockwise direction gives $\sum_{i=1}^r \pi_*\Sigma_{i*}{\Sigma_i}^*\varkappa_w \circ \operatorname{Sym}\mathbb{R}\pi_*\operatorname{res}_i$. This gives our advertised compatibility: as morphisms in the derived category,

$$(3.11) = \sum_{i=1}^r \pi_*\Sigma_{i*}{\Sigma_i}^*\varkappa_w \circ \text{natural}\circ\operatorname{Sym}\mathbb{R}\pi_*\operatorname{res}_i\,. \tag{3.16}$$

3.2.2. *Cochain-level realization.* We will need cochain-level realizations of the two compatible derived-level morphisms (3.8) and (3.14). From now on we assume that the morphism $\pi : \mathfrak{C} \to S$ is projective and fix a relatively ample line bundle $\mathcal{O}(1)$. For example, if we have a linearization θ of the G-action on V with a lift $\nu \in \widehat{\Gamma}$, we would assume that the ampleness part of the stability condition (1.3.1) holds, namely that $\mathcal{O}(1) := \omega_{\mathfrak{C}}^{\log} \otimes \mathcal{L}_\nu^{\otimes M}$ is π-ample (for some positive M).

Choose a two-term finitely generated resolution by vector bundles,

$$\mathbb{R}\pi_*\mathcal{V} \cong [A \xrightarrow{d} B] \quad \text{on } S,$$

which exists by the projectivity of π.

Definition 3.2.1. *The resolution $[A \xrightarrow{d} B]$ is called* admissible *if it satisfies Conditions 1, 2, and 3 below.*

Condition 1. *The resolution $[A \to B]$ satisfies the following:*

(a) *The map* (3.7) *is realized at the cochain level i.e. as a map*

$$A \xrightarrow{\mathrm{ev}^i_A} \pi_*(\mathcal{V}|_{\mathscr{G}_i}) \tag{3.17}$$

(b) *The map* $A \xrightarrow{\oplus_{i=1}^r \mathrm{ev}^i_A} \pi_*(\mathcal{V}|_{\mathscr{G}})$ *is surjective*

Condition 2. *There are homomorphisms* $Z : \mathrm{Sym}\, A \to \mathcal{O}_S^r$ *and* $\alpha^\vee : \mathrm{Sym}\, A \otimes B \to \mathcal{O}_S$ *making a commutative diagram*

$$\begin{tikzcd} \mathrm{Sym}\, A \arrow[r] \arrow[d, "Z"'] & \mathrm{Sym}\, A \otimes B \arrow[d, "\alpha^\vee"'] \\ \mathcal{O}_S^r \arrow[r, "\mathrm{sum}"] & \mathcal{O}_S \end{tikzcd} \tag{3.18}$$

such that the restriction

$$\begin{tikzcd} \mathrm{Sym}^d\, A \arrow[r, "(\delta, Z)"] \arrow[d] & \mathrm{Sym}^{d-1}\, A \otimes B \oplus \mathcal{O}_S^r \arrow[r] \arrow[d, "(\alpha^\vee|_{\mathrm{Sym}^{d-1}\, A\otimes B}, \mathrm{sum})"] & \mathrm{Sym}^{d-2}\, A \otimes \bigwedge^2 B \arrow[r] & \ldots \\ 0 \arrow[r] & \mathcal{O}_S & & \end{tikzcd} \tag{3.19}$$

is a cochain level realization of the map (3.14).

Condition 3. *The compatibility* (3.16) *is realized at the cochain level, i.e. the following holds:*

$$Z = \sum_{i=1}^r \pi_* \Sigma_{i*} \Sigma_i^* \varkappa_w \circ \mathrm{natural} \circ \mathrm{Sym}\, \mathrm{ev}^i_A,$$

where ev^i_A *is the map in* (3.17).

Remark 3.2.2. In what follows, when we refer to an admissible resolution $[A \xrightarrow{d} B]$, we mean the resolution, *together with choices of maps* $\mathrm{ev}^i_A, Z, \alpha^\vee$ satisfying the conditions above.

3.3. **Definition of the Polishchuk–Vaintrob factorization.** For an admissible resolution as above, let $\mathrm{tot}(A) := \mathrm{Spec}(\mathrm{Sym}\, A^\vee)$, with projection $p : \mathrm{tot}(A) \to S$.

3.3.1. *Evaluation maps.* Recall the morphism (3.5)

$$\mathrm{ev}^i_{\mathcal{V}} : \mathrm{tot}(\pi_* \mathcal{V}|_{\mathscr{G}_i}) \to I[V/G].$$

Definition 3.3.1. *The* evaluation map $\mathrm{ev}^i : \mathrm{tot}(A) \to I[V/G]$ *is the composition*

$$\mathrm{tot}(A) \xrightarrow{\mathrm{ev}^i_A} \mathrm{tot}(\pi_* \mathcal{V}|_{\mathscr{G}_i}) \xrightarrow{\mathrm{ev}^i_{\mathcal{V}}} I[V/G].$$

3.3.2. *Maps α and β.* Let

$$\beta = p^* d \circ \tau_A \colon\ \mathcal{O}_{\mathrm{tot}(A)} \to p^* A \to p^* B \tag{3.20}$$

be the section induced by the differential $A \xrightarrow{d} B$, where τ_A is the tautological section of $p^* A$ obtained by the Casimir element in $A^\vee \otimes A \subseteq \mathrm{Sym} A^\vee \otimes A$. Abusing notation, we also denote the associated cosection as

$$\alpha^\vee \colon p^* B \to \mathcal{O}_{\mathrm{tot}(A)}\,.$$

We abuse notation further and denote also by $w : I[V/G] \to \mathbb{A}^1$ the composition $I[V/G] \to [V/G] \xrightarrow{w} \mathbb{A}^1$.

Proposition 3.3.2. *Given any admissible resolution $[A \to B]$ of $\mathbb{R}\pi_*\mathcal{V}$ we have the following equality:*

$$\alpha^\vee \circ \beta = \sum_i w \circ \mathrm{ev}^i \tag{3.21}$$

Proof. By commutativity of (3.18),

$$\sum_i \alpha^\vee(a_1 \cdot ... \cdot \hat{a}_i \cdot ... \cdot a_t \otimes d(a_i)) = \mathrm{sum} \circ Z(a_1 \cdot ... \cdot a_t). \tag{3.22}$$

Setting $a_i = a$ for all i we get

$$\alpha^\vee(ta^{\otimes t-1} \otimes d(a)) = \mathrm{sum} \circ Z(a^{\otimes t}).$$

This is precisely

$$\alpha^\vee \circ \beta(1)$$

where we view the tautological section of $\mathrm{Sym}^t A$ as the map

$$a \mapsto \frac{a^{\otimes t}}{t!}$$

On the other hand by Condition 3, the right-hand-side of (3.22) is

$$\sum_i \langle (\mathrm{Sym}\ \mathrm{ev}^i_A)^\vee \circ \mathrm{natural}^\vee \circ (\pi_* \Sigma_{i*} \Sigma_i^* \varkappa_w)^\vee(1), a_1 \cdot ... \cdot a_t \rangle,$$

where $(\mathrm{Sym}\ \mathrm{ev}^i_A)^\vee \circ \mathrm{natural}^\vee \circ (\pi_* \Sigma_{i*} \Sigma_i^* \varkappa_w)^\vee$ is the composition of $\mathcal{O}_S$-module homomorphisms

$$\mathcal{O}_S \to \pi_* \mathrm{Sym}((\mathcal{V}|_{\mathscr{G}_i})^\vee) \to \mathrm{Sym}((\pi_* \mathcal{V}|_{\mathscr{G}_i})^\vee) \to \mathrm{Sym}(A^\vee),$$

which coincides with the homomorphism $\mathcal{O}_S \to \mathrm{Sym}(A^\vee)$ associated to the map $w \circ \mathrm{ev}^i$.

□

Remark 3.3.3. In [PV16, Section 4.1], they rescale the map $\alpha^\vee$. This is, roughly, due to the fact that they view the tautological section of $\mathrm{Sym}^t A$ as the map $a \mapsto a^{\otimes t}$ without the factor of $t!$. More precisely, they assemble $\alpha^\vee$ from pieces obtained from each monomial in w using tautological sections of the corresponding monomial component of $\mathrm{Sym}\, A$ using the R-charge decomposition $A = A_1 \oplus ... \oplus A_n$ (see §3.7). Our rescaling of the tautological section is consistent with viewing the exponential function on the tautological section for A as a tautological section of $\mathrm{Sym}\, A$ which is what we do in §3.6.3.

Definition 3.3.4. *Let $[A \to B]$ be an admissible resolution of $\mathbb{R}\pi_*\mathcal{V}$ on S. The Polishchuk–Vaintrob (PV) factorization associated to $(\mathfrak{C} \to S, P, V, \Gamma, \chi, \varkappa, w)$ is the Koszul factorization $\{-\alpha, \beta\}$ on $\mathrm{tot}\, A$.*

3.4. **On Condition 1.** We show that resolutions satisfying Condition 1 can be constructed over an arbitrary base S.

Lemma 3.4.1. *There is a π-acyclic resolution $[\mathcal{A} \to \mathcal{B}]$ of $\mathcal{V}$ on $\mathfrak{C}$, with $\mathcal{A}, \mathcal{B}$ locally free of finite rank, and with a homomorphism $\mathcal{A} \to \mathcal{V}|_{\mathscr{G}}$ satisfying:*

(a) *The diagram*

$$\begin{tikzcd} \mathcal{V} \arrow[r] \arrow[d] & \mathcal{A} \arrow[d] \\ \mathcal{V}|_{\mathscr{G}} \arrow[r, "="'] & \mathcal{V}|_{\mathscr{G}} \end{tikzcd}$$

is commutative.

(b) *The kernel of $\mathcal{A} \to \mathcal{V}|_{\mathscr{G}}$ is locally free and π-acyclic.*

Proof. Using the π-ample line bundle $\mathcal{O}(1)$, we may choose a finitely generated locally free resolution of $\mathcal{V}$

$$0 \to \mathcal{V} \to \mathcal{A}' \to \mathcal{B}' \to 0. \tag{3.23}$$

such that $\mathcal{A}'(-\mathscr{G})$ is π-acyclic. Composing the restriction $\mathcal{A}' \to \mathcal{A}'|_{\mathscr{G}}$ with the first projection, we get a map $f : \mathcal{A}' \oplus \mathcal{V}|_{\mathscr{G}} \to \mathcal{A}'|_{\mathscr{G}}$. Similarly, we have another map $g : \mathcal{A}' \oplus \mathcal{V}|_{\mathscr{G}} \to \mathcal{A}'|_{\mathscr{G}}$, obtained by composing the inclusion $\mathcal{V}|_{\mathscr{G}} \to \mathcal{A}'|_{\mathscr{G}}$ with the second projection. Define $\mathcal{A}$ as the equalizer

$$\begin{tikzcd} \mathcal{A} \arrow[r] & \mathcal{A}' \oplus \mathcal{V}|_{\mathscr{G}} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathcal{A}'|_{\mathscr{G}}. \end{tikzcd}$$

Note that the kernel of the projection $\mathcal{A} \to \mathcal{V}|_{\mathscr{G}}$ is canonically isomorphic to $\mathcal{A}'(-\mathscr{G})$. Since $\mathcal{A}'(-\mathscr{G})$ is π-acyclic, so is $\mathcal{A}$.

Now we have a natural monomorphism $\mathcal{V} \to \mathcal{A}$ and we define $\mathcal{B}$ as its cokernel. Since the exact sequence (3.23) is locally split, one can easily check that $\mathcal{A}$, $\mathcal{B}$ are finitely generated locally free. $\square$

Corollary 3.4.2. *For $\mathcal{A}, \mathcal{B}$ as in Lemma* (3.4.1)*, we obtain a two term vector bundle resolution*

$$\mathbb{R}\pi_*\mathcal{V} \cong [\pi_*\mathcal{A} \to \pi_*\mathcal{B}]$$

with a surjective homomorphism $\pi_*\mathcal{A} \to \oplus_i \pi_*(\mathcal{V}|_{\mathscr{G}_i})$.

The following lemma will be used several times in later sections.

Lemma 3.4.3. *Suppose $[A' \to B']$ is a resolution of $\mathbb{R}\pi_*\mathcal{V}$ satisfying Condition 1 (respectively 2, 3) and*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "e_A"] & B \arrow[d, "e_B"] \\ A' \arrow[r] & B' \end{tikzcd}$$

is a quasi-isomorphism. Assume that e_B is surjective. Then $[A \to B]$ is a resolution of $\mathbb{R}\pi_*\mathcal{V}$ satisfying Condition 1 (respectively 2, 3).

Proof. For Condition 1, define ev^i_A as the following composition

$$\mathrm{ev}^i_A : A \xrightarrow{e_A} A' \xrightarrow{\mathrm{ev}^i_{A'}} \mathcal{O}_U$$

which clearly satisfies Condition 1 if $\mathrm{ev}^i_{A'}$ does. For Conditions 2, 3, define Z and $\alpha^\vee$ as the following compositions

$$Z : \mathrm{Sym}\,\bar{A} \xrightarrow{\mathrm{Sym}(e_A)} \mathrm{Sym}\,\widetilde{A} \xrightarrow{Z'} \mathcal{O}_U\,.$$

$$\alpha^\vee : \mathrm{Sym}\,\bar{A} \otimes \bar{B} \xrightarrow{\mathrm{Sym}(e_A)\otimes e_B} \mathrm{Sym}\,A' \otimes B' \xrightarrow{(\alpha')^\vee} \mathcal{O}_U\,.$$

where $(\alpha')^\vee$ and Z' are the corresponding maps for the resolution $[A' \to B']$. Condition 2 for $Z, \alpha^\vee$ is immediate from Condition 2 for $Z', (\alpha')^\vee$. Condition 3 for Z follows from Condition 3 for Z' using functoriality of Sym. $\square$

3.5. **On Conditions 2 and 3.** In this subsection, our goal is to investigate when admissible resolutions exist on a stack $S \to \mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma)_{\omega^{\log}_{\mathfrak{C}}}$. We explain how an argument of Polishchuk and Vaintrob [PV16] can be adapted to provide an explicit construction. This argument requires a certain cohomology vanishing statement and unfortunately we are able to ensure it holds only after imposing some restrictions on S. Therefore, in this subsection we will make the following:

Assumption $(\star)$: S is a Deligne–Mumford stack of finite type over $\mathrm{Spec}(\mathbb{C})$, which is a global quotient stack by a linear algebraic group action, and whose coarse moduli space is projective over an affine noetherian scheme.

Remark 3.5.1. Some of the requirements in Assumption $(\star)$ are relatively harmless for the applications we envision to GLSM theory. Typically S will be an appropriate moduli stack of LG quasimaps and the finite type and Deligne–Mumford condition will be automatic once a stability condition is imposed. Also, these moduli stacks are all expected to be global quotients and there are well-known techniques to prove this is indeed the case in many situations. However, projectivity over an affine of the coarse moduli is not automatic.

Proposition 3.5.2. *Let U be a stack equipped with a morphism $U \xrightarrow{u} \mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma)_{\omega^{\log}_{\mathfrak{C}}}$. If u factors through a morphism $S \to \mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma)_{\omega^{\log}_{\mathfrak{C}}}$, with S a stack satisfying Assumption $(\star)$ (for example, if U is an open substack of S), then there exists an admissible resolution $[A \to B]$ of $\mathbb{R}(\pi_U)_*\mathcal{V}$ on U.*

Proof. First notice that if $[A \to B]$ is an admissible resolution on S, then its pullback is an admissible resolution on U. Hence we may assume $U = S$.

Two technical lemmas are required for the proof of Proposition 3.5.2. The first lemma is a slightly more general version of [PV16, Lemma 4.2.4], which is in fact what their argument establishes. The second lemma is a corollary of [PV16, Lemma 4.2.5], as we explain.

Lemma 3.5.3. *Let S be a stack satisfying Assumption $(\star)$. Let $\mathcal{O}(1)$ denote the pull-back to S of a relatively ample line bundle on the coarse moduli $\underline{S}$. Then, there exists a vector bundle $\mathcal{E}$ on S such that for any coherent sheaf $\mathcal{F}$ on S the natural map*

$$\mathrm{H}^0(S, \mathcal{E}^\vee \otimes \mathcal{F}(n)) \otimes \mathcal{E}(-n) \to \mathcal{F}$$

is surjective for $n >> 0$ and

$$\mathrm{H}^i(S, \mathcal{F}(n)) = 0$$

for $i > 0, n >> 0$.

Lemma 3.5.4. *Let S be a stack satisfying Assumption ($\star$). Let $\mathcal{E}$ be a vector bundle on S satisfying the conclusion of Lemma 3.5.3.*

Let $[C_0 \to C_1]$ be a 2-term complex of vector bundles on S. Given the above setup, a positive integer $d \in \mathbb{N}$, and any vector bundle $\mathcal{D}$ on S, there exists $m_0 > 0$ such that for any $m_1 \geq m_0$ and any surjection

$$\bar{C}_1 := \mathcal{E}^\vee(-m_1)^{\oplus N} \xrightarrow{\sigma} C_1 \to 0$$

one has

$$\mathrm{H}^i(S, (\mathrm{Sym}^{q_1} \bar{C}_0)^\vee \otimes \bigwedge\nolimits^{q_2} \bar{C}_1^\vee \otimes \mathcal{D}) = 0 \tag{3.24}$$

for $d \geq q_1 + q_2$, $q_2 \geq 1$, where the bundle $\bar{C}_0$ is the fiber product of C_0 and $\bar{C}_1$ over C_1, which completes the following vertical quasi-isomorphism,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bar{C}_0 & \longrightarrow & \bar{C}_1 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_0 & \longrightarrow & C_1 & \longrightarrow & 0. \end{array}$$

Proof. Given $[C_0 \to C_1]$ and $\mathcal{D}$ as above, the statement of the lemma is the same as the statement of the lemma for the complex $[C_0 \oplus \mathcal{D} \to C_1]$ where, instead of the original $\mathcal{D}$, we now have $\mathcal{D} = \mathcal{O}$. Hence, we may assume $\mathcal{D} = \mathcal{O}_S$. The vanishing (3.24) now follows from Equation (4.26) of [PV16, Lemma 4.2.5], in view of the fact that, since we work in characteristic zero, for any vector bundle $\mathcal{W}$, both the symmetric power $\mathrm{Sym}^q \mathcal{W}$ and the exterior power $\bigwedge^q \mathcal{W}$ are direct summands in $\mathcal{W}^{\otimes q}$. □

We now return to the proof of Proposition 3.5.2. Start with a resolution $\pi_*\mathcal{A} \to \pi_*\mathcal{B}$ satisfying Condition 1, whose existence is guaranteed by Corollary 3.4.2. We modify it so that it also satisfies Condition 2. Specifically, by Lemma 3.5.4, we may choose a component-wise surjective quasi-isomorphism

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow{\scriptstyle e_A} & & \downarrow{\scriptstyle e_B} \\ \pi_*\mathcal{A} & \longrightarrow & \pi_*\mathcal{B} \end{array}$$

such that

$$\mathrm{Ext}^q(\mathrm{Sym}^i A \otimes \bigwedge\nolimits^j B, \mathcal{O}_S) = 0 \tag{3.25}$$

for $i + j \leq d_0$, $j \geq 1$ and $q \geq 1$. Here d_0 is the polynomial degree of w so that $w \in \mathrm{Sym}^{\bullet \leq d_0} V^\vee$. Now, we may define

$$Z := \sum_{i=1}^{r} \pi_* \Sigma_{i*} \Sigma_i^* \varkappa_w \circ \mathrm{natural} \circ \mathrm{Sym}\, \mathrm{ev}_A^i$$

Then, we may take E_d to be the complex

$$E_d := 0 \longrightarrow \mathrm{Sym}^d A \xrightarrow{(\delta, Z)} \mathrm{Sym}^{d-1} A \otimes B \oplus \mathcal{O}_S^r \xrightarrow{(\delta, 0)} \mathrm{Sym}^{d-2} A \otimes \bigwedge\nolimits^2 B \xrightarrow{\delta} \dots$$

There is a spectral sequence

$$E_1^{p,q} = \bigoplus_p \mathrm{Ext}^q(E_d^p, \mathcal{O}_S) \Rightarrow \mathrm{Hom}_{\mathrm{D}(S)}(E_d^p, \mathcal{O}_S[p+q]).$$

where E_d^p denotes the degree p component of E_d. It follows from (3.25) that for $p+q=-1$ and every $0 \le d \le d_0$, the morphism (3.14) can be realized at the cochain level (see e.g. the proof of Lemma 5.7 on page 89 of [GKZ08]). That is, for $d \le d_0$ there exists $\alpha_d^\vee$ realizing (3.14) as follows

$$\begin{tikzcd} \operatorname{Sym}^d A \arrow[r, "{(\delta,Z)}"] \arrow[d] & \operatorname{Sym}^{d-1} A \otimes B \oplus \mathcal{O}_S^r \arrow[r, "{(\delta,0)}"] \arrow[d, "{(\alpha_d^\vee,\mathrm{sum})}"] & \operatorname{Sym}^{d-2} A \otimes \bigwedge^2 B \arrow[r, "\delta"] \arrow[d] & \dots \\ 0 \arrow[r] & \mathcal{O}_S \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd}$$

Then we can define

$$\alpha^\vee := \sum_{d=0}^{d_0} \alpha_d^\vee.$$

Conditions 2 and 3 are manifestly satisfied by this construction.

□

Proposition 3.5.2 applies in particular to the case of the hybrid models of §1.4. However, for the analysis of the geometric phase in §6.1, we will need a refinement in which the admissible resolution satisfies an additional property. This is spelled out explicitly in the following statement.

Proposition 3.5.5. *Assume we have a hybrid model, so that $V = V_1 \oplus V_2$. Assume further that the potential is linear on V_2, i.e., that $w \in V_2^\vee \otimes \operatorname{Sym}^{\ge 1}(V_1^\vee)$. Suppose $U \xrightarrow{u} \mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma)_{\omega^{\log}_{\mathfrak{C}}}$ factors through a stack S over $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma)_{\omega^{\log}_{\mathfrak{C}}}$ satisfying Assumption ($\star$). Then we may choose an admissible resolution $[A \to B]$ of $\mathbb{R}\pi_*\mathcal{V}$ on U with $A = A_1 \oplus A_2' \oplus \pi_*(\mathcal{V}_2|_{\mathscr{G}})$, and such that ev^i_A followed by the projection onto $\pi_*(\mathcal{V}_2|_{\mathscr{G}})$ agrees with the projection of A onto this summand.*

Proof. We use the direct sum decomposition $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ to resolve the summands individually.

First we find a resolution $[\pi_*\mathcal{A}_1 \to \pi_*\mathcal{B}_1]$ of $\mathbb{R}\pi_*\mathcal{V}_1$ as in Corollary 3.4.2.

Next, choose any π-acyclic locally free resolution of $\mathcal{V}_2$ on $\mathfrak{C}$,

$$0 \to \mathcal{V}_2 \to \mathcal{A}_2' \to \mathcal{B}_2' \to 0.$$

and define $\mathcal{B}_2$ to be the cokernel of the map $\mathcal{V}_2 \to \mathcal{A}_2' \oplus \mathcal{V}_2|_{\mathscr{G}}$. The snake lemma gives the diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathcal{V}_2(-\mathscr{G}) \arrow[r] \arrow[d] & \mathcal{A}_2' \arrow[r] \arrow[d] & \mathcal{B}_2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{V}_2 \arrow[r] \arrow[d] & \mathcal{A}_2' \oplus \mathcal{V}_2|_{\mathscr{G}} \arrow[r] \arrow[d] & \mathcal{B}_2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{V}_2|_{\mathscr{G}} \arrow[r] \arrow[d] & \mathcal{V}_2|_{\mathscr{G}} \arrow[r] \arrow[d] & 0 & \\ & 0 & 0 & & \end{tikzcd} \tag{3.26}$$

Pushing forward via π, we get a resolution $\mathbb{R}\pi_*\mathcal{V}_2 \cong [(\pi_*\mathcal{A}'_2 \oplus \pi_*\mathcal{V}_2|_{\mathscr{G}}) \to \pi_*\mathcal{B}_2]$. Set

$$A_2 := \pi_*\mathcal{A}'_2 \oplus \pi_*\mathcal{V}_2|_{\mathscr{G}}, \quad B_2 := \pi_*\mathcal{B}_2.$$

Then

$$[\pi_*\mathcal{A}_1 \oplus A_2 \to \pi_*\mathcal{B}_1 \oplus B_2] \cong \mathbb{R}\pi_*\mathcal{V}$$

manifestly satisfies Condition 1.

The morphism $\mathrm{Sym}[\pi_*\mathcal{A}_1 \oplus A_2 \to \pi_*\mathcal{B}_1 \oplus B_2] \to [\mathcal{O}_S^r \to \mathcal{O}_S]$ factors through the projection

$$\mathrm{Sym}[\pi_*\mathcal{A}_1 \oplus A_2 \to \pi_*\mathcal{B}_1 \oplus B_2] \to [A_2 \to B_2] \otimes \mathrm{Sym}[\pi_*\mathcal{A}_1 \to \pi_*\mathcal{B}_1].$$

Now we use Lemma 3.5.4 to modify only the first summands, by choosing a quasi-isomorphism

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "e_A"] & B_1 \arrow[d, "e_B"] \\ \pi_*\mathcal{A}_1 \arrow[r] & \pi_*\mathcal{B}_1 \end{tikzcd}$$

such that $\mathrm{Ext}^q(\mathrm{Sym}^i A_1 \otimes \bigwedge^j B_1 \otimes A_2, \mathcal{O}_S) = 0$, $\mathrm{Ext}^q(\mathrm{Sym}^i A_1 \otimes \bigwedge^j B_1 \otimes B_2, \mathcal{O}_S) = 0$ for $i+j \leq d_0 - 1$, $j \geq 1$, and $q \geq 1$. The resolution $[A_1 \oplus A_2 \to B_1 \oplus B_2]$ is admissible and satisfies the required property, with $A' := A_1 \oplus \pi_*\mathcal{A}'_2$. □

3.6. **Support of the PV factorization.**

3.6.1. *Base change and* $\mathrm{tot}\,\pi_*(\mathcal{V})$. Denote by

$$\mathrm{tot}(\pi_*\mathcal{V}) := \mathrm{Spec}\,\mathrm{Sym}\,\mathbb{R}^1\pi_*(\mathcal{V}^\vee \otimes \omega_{\mathfrak{C}}),$$

so that, for any $g\colon T \to S$, $\mathrm{Hom}_S(T, \mathrm{tot}(\pi_*\mathcal{V})) = \Gamma(\mathfrak{C}_T, \mathcal{V}_T)$ by the canonical Serre duality identification $\mathbb{R}^1\pi_*(\mathcal{V}^\vee \otimes \omega_{\mathfrak{C}})^\vee \cong \pi_*\mathcal{V}$. Here $\mathcal{V}_T := h^*\mathcal{V}$, from the pullback diagram

$$\begin{tikzcd} \mathfrak{C}_T \arrow[r, "h"] \arrow[d, "\pi_T"] & \mathfrak{C} \arrow[d, "\pi"] \\ T \arrow[r, "g"] & S. \end{tikzcd} \tag{3.27}$$

Note that since π is flat of relative dimension one, Tor-independent base change gives an isomorphism $\mathbb{R}^1(\pi_T)_*h^*\mathcal{E} \cong g^*\mathbb{R}^1\pi_*\mathcal{E}$ for any vector bundle on $\mathfrak{C}$. Hence $\mathrm{tot}((\pi_T)_*h^*\mathcal{V}) \cong \mathrm{tot}(\pi_*\mathcal{V}) \times_S T$.

3.6.2. *The map* dw_T. Considering w as a Γ-invariant element of $\mathrm{Sym}V^\vee \otimes \mathbb{C}_\chi$, we obtain its differential $dw \in \mathrm{Sym}V^\vee \otimes V^\vee \otimes \mathbb{C}_\chi$ which is Γ-invariant. Recall that there is a natural pairing

$$\mathrm{Sym}\,V \times \mathrm{Sym}\,V^\vee \to k$$
$$(v_1 \otimes \dots \otimes v_n, \phi_1 \otimes \dots \otimes \phi_n) \mapsto \sum_{\sigma \in S_n} \prod_{i=1}^n \phi_i(v_{\sigma(i)}).$$

This leads to a natural isomorphism in characteristic 0 which we describe on a basis

$$\begin{aligned} (\mathrm{Sym}\,V)^\vee &\to (\mathrm{Sym}\,V)^\vee \\ e_1 \otimes \dots \otimes e_n &\mapsto n! e_1^* \otimes \dots \otimes e_n^* \end{aligned} \tag{3.28}$$

Under this convention, pairing with dw gives a map of Γ-representations

$$\overline{dw}\colon \mathrm{Sym}\,V \to V^\vee \otimes \mathbb{C}_\chi.$$

For any $T \xrightarrow{g} S$, with T a scheme, it induces a homomorphism $\mathcal{O}_S$-modules

$$\overline{dw}_T : \pi_*(\operatorname{Sym} \mathcal{V}_T) \to \pi_*(\mathcal{V}_T^\vee \otimes \omega_{\mathfrak{C}}^{\log}),$$

via pull-back by $[g^*P] : \mathfrak{C}_T \to B\Gamma$, followed by push-forward by $\pi = \pi_T$. The composition

$$\operatorname{Sym} \pi_*(\mathcal{V}_T) \xrightarrow{\text{natural}} \pi_*(\operatorname{Sym} \mathcal{V}_T) \xrightarrow{\overline{dw}_T} \pi_*(\mathcal{V}_T^\vee \otimes \omega_{\mathfrak{C}}^{\log}) \cong (\mathbb{R}^1\pi_*\mathcal{V}_T(-\mathscr{G}))^\vee$$

(where the last isomorphism comes from Serre duality) induces the homomorphism

$$dw_T : \operatorname{Sym} \pi_*(\mathcal{V}_T) \otimes \mathbb{R}^1\pi_*(\mathcal{V}_T(-\mathscr{G})) \to \mathcal{O}_T. \tag{3.29}$$

This can be alternatively described as the map

$$\operatorname{Sym} \pi_*(\mathcal{V}_T) \otimes \mathbb{R}^1\pi_*(\mathcal{V}_T(-\mathscr{G})) \to \mathbb{R}^1\pi_*(\omega_{\mathfrak{C}}) \cong \mathcal{O}_T.$$

induced by $\overline{dw}$ as in Section 3.2 of [CL11].

3.6.3. *The degeneracy locus $Z(dw_{\operatorname{tot}(\pi_*\mathcal{V})}(\exp \tau_{\pi_*\mathcal{V}}))$.* Consider the map (3.29) associated to the projection $p' : \operatorname{tot}(\pi_*\mathcal{V}) \to S$. Consider also the tautological global section $\tau_{\pi_*\mathcal{V}}$ of $\pi_*(\mathcal{V}_{\operatorname{tot}(\pi_*\mathcal{V})})$. Together, they determine a cosection of the sheaf $\mathbb{R}^1\pi_*\mathcal{V}_{\operatorname{tot}(\pi_*\mathcal{V})}(-\mathscr{G})$,

$$dw_{\operatorname{tot}(\pi_*\mathcal{V})}(\exp \tau_{\pi_*\mathcal{V}}) : \mathbb{R}^1\pi_*\mathcal{V}_{\operatorname{tot}(\pi_*\mathcal{V})}(-\mathscr{G}) \to \mathcal{O}_{\operatorname{tot}(\pi_*\mathcal{V})}. \tag{3.30}$$

A priori, the exponential in the formula is an element of the completed symmetric algebra, $\exp \tau_{\pi_*\mathcal{V}} \in \widehat{\operatorname{Sym}} H^0(\operatorname{tot}(\pi_*\mathcal{V}), \pi_*(\mathcal{V}_{\operatorname{tot}(\pi_*\mathcal{V})}))$, but we view it as an element of $\operatorname{Sym}(\pi_*(\mathcal{V}_{\operatorname{tot}(\pi_*\mathcal{V})})$ by first considering its truncation modulo $(\tau_{\pi_*\mathcal{V}})^{m+1}$ and then taking the image under the natural map

$$\operatorname{Sym}^{\leq m} H^0(\operatorname{tot}(\pi_*\mathcal{V}), \pi_*(\mathcal{V}_{\operatorname{tot}(\pi_*\mathcal{V})})) \otimes \mathcal{O}_{\operatorname{tot}(\pi_*\mathcal{V})} \longrightarrow \operatorname{Sym}^{\leq m}(\pi_*(\mathcal{V}_{\operatorname{tot}(\pi_*\mathcal{V})})).$$

In this way, we view $\exp \tau_{\pi_*\mathcal{V}}$ the tautological section of a truncation of $\operatorname{Sym}(\pi_*(\mathcal{V}_{\operatorname{tot}(\pi_*\mathcal{V})})$, which is consistent with earlier conventions (see Remark 3.3.3).

We denote by

$$Z(dw_{\operatorname{tot}(\pi_*\mathcal{V})}(\exp \tau_{\pi_*\mathcal{V}})) := \operatorname{Spec}(\operatorname{coker} dw_{\operatorname{tot}(\pi_*\mathcal{V})}(\exp \tau_{\pi_*\mathcal{V}})) \subseteq \operatorname{tot}(\pi_*\mathcal{V})$$

the degeneracy locus of the cosection (3.30).

Proposition 3.6.1. *Assume that there is an admissible resolution $[A \to B]$ of $\mathbb{R}\pi_*\mathcal{V}$ on S. Then $Z(dw_{\operatorname{tot}(\pi_*\mathcal{V})}(\exp \tau_{\pi_*\mathcal{V}}))$ is canonically identified with a closed substack of* $\operatorname{tot} A$ *and the PV factorization $\{-\alpha, \beta\}$ is supported on $Z(dw_{\operatorname{tot}(\pi_*\mathcal{V})}(\exp \tau_{\pi_*\mathcal{V}}))$.*

The rest of this subsection will be occupied by the proof of the above Proposition. Since by Proposition 2.3.3 a Koszul factorization is always supported on the common zero locus $Z(\alpha, \beta)$, we begin with describing $Z(\beta)$ and $Z(\alpha)$ for the PV factorization.

3.6.4. *The substack $Z(\beta)$.* Let $[A \xrightarrow{d} B]$ be a two-term vector bundle complex with an isomorphism φ between $[A \to B]$ and $\mathbb{R}\pi_*\mathcal{V}$ in $\mathrm{D}(S)$. Let $p : \operatorname{tot} A \to S$ be the projection and let $\beta : \mathcal{O}_{\operatorname{tot} A} \to p^*B$ be the section induced by d, as in (3.20). Denote by $Z(\beta) \subseteq \operatorname{tot} A$ its zero locus. The following lemma establishes in particular the first assertion of Proposition 3.6.1.

Lemma 3.6.2. *There is an isomorphism $Z(\beta) \cong \operatorname{tot} \pi_*\mathcal{V}$, induced by φ.*

Proof. First we note that $Z(\beta)$ is the closed substack of $\operatorname{tot} A$ parameterizing objects $f : T \to \operatorname{tot} A$ for which $f^*\beta : \mathcal{O}_T \to g^*B$ is the zero map, where $g = p \circ f : T \to S$.

On the other hand, if we start with $g : T \to S$, to give a lift $f : T \to \operatorname{tot} A$ amounts to giving a global section $\sigma_f \in H^0(T, g^*A)$, with $\sigma_f = f^*\tau_A$.

We have $f^*\beta = (g^*d) \circ \sigma_f$, so it vanishes identically if and only if σ_f factors through the kernel of $g^*A \xrightarrow{g^*d} g^*B$. The composition

$$\mathcal{O}_T \xrightarrow{\sigma_f} \ker(g^*d) \xrightarrow[\sim]{h^0(\varphi)} (\pi_T)_*(\mathcal{V}_T)$$

gives the required object of $\operatorname{tot} \pi_*\mathcal{V}$. □

3.6.5. *The substack* $Z(\alpha)$. We keep the set-up from §3.6.4, and assume further that the resolution $[A \xrightarrow{d} B] \overset{\varphi}{\cong} \mathbb{R}\pi_*\mathcal{V}$ is admissible, so that we have the cosection $\alpha^\vee : p^*B \to \mathcal{O}_{\operatorname{tot} A}$.

Its degeneracy locus $Z(\alpha)$ (which is the same as the zero locus of the dual section of $p^*B^\vee$) parametrizes the closed substack of $\operatorname{tot} A$ whose objects are maps $f : T \to \operatorname{tot} A$, T a scheme, for which $f^*\alpha^\vee = 0$. Note that $f^*\alpha^\vee$ is the $\mathcal{O}_T$-module homomorphism $B_T \to \mathcal{O}_T$ given by

$$f^*(\alpha^\vee)(b) = \alpha_T^\vee((\exp \sigma_f) \otimes b)$$

for every local section b of the vector bundle $B_T := g^*B$ on T. Here $\alpha_T^\vee := g^*(\alpha^\vee)$, and we interpret $\exp \sigma_f$ as the local restriction of a suitable truncation (depending only on w) of an element in $\widehat{\operatorname{Sym}} H^0(T, A_T)$, as explained in §3.6.3.

As $[A \xrightarrow{d} B]$ is admissible, it satisfies Condition 1 of Definition 3.2.1. Define the vector bundle A' on S as the kernel in the exact sequence

$$0 \to A' \to A \to \pi_*(\mathcal{V}|_{\mathscr{G}}) \to 0.$$

The quasi-isomorphism φ induces a quasi-isomorphism

$$[A' \xrightarrow{d'} B] \overset{\varphi'}{\cong} \mathbb{R}\pi_*\mathcal{V}(-\mathscr{G}),$$

where $d' = d|_{A'}$. Hence there is a surjective "connecting homomorphism"

$$\partial : B \twoheadrightarrow \operatorname{coker} d' \xrightarrow[\sim]{h^1(\varphi')} \mathbb{R}^1\pi_*\mathcal{V}(-\mathscr{G}).$$

Moreover, for any $T \xrightarrow{g} S$, we have a corresponding $\partial : B_T \twoheadrightarrow \mathbb{R}^1\pi_*\mathcal{V}_T(-\mathscr{G})$ by the base-change property of $\mathbb{R}^1\pi_*\mathcal{V}(-\mathscr{G})$.

Lemma 3.6.3. *Given* $T \xrightarrow{g} S$*, the composition*

$$\operatorname{Sym}(\pi_*\mathcal{V}_T) \otimes B_T \xrightarrow{\operatorname{id} \otimes \partial} \operatorname{Sym}(\pi_*\mathcal{V}_T) \otimes \mathbb{R}^1\pi_*\mathcal{V}_T(-\mathscr{G}) \xrightarrow{dw_T} \mathcal{O}_T$$

coincides with the restriction of $\alpha_T^\vee$ *to* $\operatorname{Sym}(\pi_*\mathcal{V}_T) \otimes B_T$*.*

Granting the lemma, we can easily finish the proof of Proposition 3.6.1. Indeed, when f factors as

$$T \to \operatorname{tot} \pi_*\mathcal{V} = Z(\beta) \subseteq \operatorname{tot} A,$$

we have $\sigma_f = f^*(\tau_{\pi_*\mathcal{V}})$. By Lemma 3.6.3,

$$\alpha_T^\vee((\exp \sigma_f) \otimes b) = 0 \iff dw_T((\exp \sigma_f) \otimes \partial b) = 0$$

for every local section b of the vector bundle B_T on T. By the surjectivity of ∂, we conclude that $Z(\alpha) \cap Z(\beta) = Z(dw_{\operatorname{tot}(\pi_*\mathcal{V})}(\exp \tau_{\pi_*\mathcal{V}}))$, as required.

3.6.6. *Proof of Lemma 3.6.3.* Let T be a scheme over S. Since we will work exclusively over T, we drop the subscript T from the curve $\mathfrak{C}_T$, the bundles $\mathcal{V}_T$, B_T, etc.

Consider the morphism (3.11) and recall that E is the shifted cone in $\mathrm{D}(T)$

$$E := \mathrm{Cone}(\mathrm{Sym}\,\mathbb{R}\pi_*\mathcal{V} \to \mathcal{O}_T^r)[-1]$$

and that (3.14) provides a morphism $E \to \mathcal{O}_T[-1]$ in $\mathrm{D}(T)$. Denote by $H^1(E \to \mathcal{O}_T)$ the induced morphism $H^1(E) \to \mathcal{O}_T$ on cohomology sheaves.

Given the resolution $[A \to B]$ of Lemma 3.6.3, there is a natural map

$$\psi : \mathrm{Sym}(\pi_*\mathcal{V}) \otimes B \to H^1(E) \tag{3.31}$$

defined as follows. Represent E as the complex

$$[\mathrm{Sym}\,A \xrightarrow{d_1^E} (\mathrm{Sym}\,A \otimes B) \oplus \mathcal{O}_S^r \xrightarrow{d_2^E} \dots]$$

and note that the inclusion

$$\mathrm{Sym}(\pi_*\mathcal{V}) \otimes B \subseteq \mathrm{Sym}\,A \otimes B \subseteq (\mathrm{Sym}\,A \otimes B) \oplus \mathcal{O}_S^r$$

factors through $\ker d_2^E$. Composing with the quotient map $\ker d_2^E \to H^1(E)$ gives the sheaf homomorphism ψ, independent of the choice of representative for E.

One easily checks that the composition $H^1(E \to \mathcal{O}_T) \circ \psi$ is precisely the restriction of $\alpha^\vee$ to $\mathrm{Sym}(\pi_*\mathcal{V}) \otimes B$. We are therefore reduced to proving that the diagram

$$\begin{tikzcd}
\mathrm{Sym}(\pi_*\mathcal{V}) \otimes B \arrow[r, "\psi"] \arrow[d, "\mathrm{Id}\otimes\partial"'] & H^1(E) \arrow[d, "H^1(E\to\mathcal{O}_T)"] \\
\mathrm{Sym}(\pi_*\mathcal{V}) \otimes \mathbb{R}^1\pi_*\mathcal{V}(-\mathscr{G}) \arrow[r, "dw_T"'] & \mathcal{O}_T
\end{tikzcd} \tag{3.32}$$

is commutative.

We prove this, by first arguing that this only needs to be done for a particular resolution. We do so below and provide the setup for a particular resolution which we show in Lemma 3.6.4 satisfies (3.32). In so doing, we complete the proof of Proposition 3.6.1.

Indeed, the maps ∂ and ψ, can be defined in the same way for *any* resolution which satisfies Condition 1 (but which is not necessarily admissible). We claim that if there exists one particular resolution $[A_0 \to B_0]$ of $\mathbb{R}^1\pi_*\mathcal{V}$ which satisfies Condition 1 and for which the diagram (3.32) commutes, then the corresponding diagram commutes for any other resolution $[A \to B]$ satisfying Condition 1. Indeed, by Lemma 3.6.5 (proven below) there exists a resolution $[\widetilde{A} \to \widetilde{B}]$, together with a *surjective* quasi-isomorphism $[\widetilde{A} \to \widetilde{B}] \to [A \to B]$, and a *surjective* quasi-isomorphism $[\widetilde{A} \to \widetilde{B}] \to [A_0 \to B_0]$. This induces the following commuting diagram

$$\begin{tikzcd}
 & \widetilde{B} \arrow[dl, two heads] \arrow[d, two heads, "\partial"] \arrow[dr, two heads] & \\
B_0 \arrow[r, two heads, "\partial"] & \mathbb{R}^1\pi_*\mathcal{V}(-\mathscr{G}) & B \arrow[l, two heads, "\partial"']
\end{tikzcd}$$

From these, the claim follows immediately. It remains to construct one resolution for which (3.32) commutes.

Let $0 \to \mathcal{V} \to [\mathcal{A} \to \mathcal{B}] \to 0$ be a resolution of $\mathcal{V}$ by vector bundles such that $\mathbb{R}^1\pi_*\mathcal{A} = \mathbb{R}^1\pi_*\mathcal{B} = 0$. Assume further that the map $\mathcal{V} \to \mathcal{V}|_{\mathscr{G}}$ extends to a map $\mathcal{A} \to \mathcal{V}|_{\mathscr{G}}$, with π-acyclic kernel. By Lemma 3.4.1 such a resolution always exists, and $[\pi_*\mathcal{A} \to \pi_*\mathcal{B}]$ is a resolution of $\mathbb{R}\pi_*\mathcal{V}$ which satisfies Condition 1.

Let $\mathcal{A}'$ be the kernel of the map $\mathcal{A} \to \mathcal{V}|_{\mathscr{G}}$. Then, the snake lemma gives the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{V}(-\mathscr{G}) \arrow[r] \arrow[d] & \mathcal{A}' \arrow[r] \arrow[d] & \mathcal{B} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{V} \arrow[r] \arrow[d] & \mathcal{A} \arrow[r] \arrow[d] & \mathcal{B} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{V}|_{\mathscr{G}} \arrow[r] \arrow[d] & \mathcal{V}|_{\mathscr{G}} \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd} \tag{3.33}$$

Hence the short exact sequence

$$0 \to \mathcal{V}(-\mathscr{G}) \to \mathcal{A}' \to \mathcal{B} \to 0$$

induces the connecting homomorphism,

$$\partial\colon \pi_*\mathcal{B} \to \mathbb{R}^1\pi_*\mathcal{V}(-\mathscr{G}).$$

Note that ∂ is surjective by the vanishing $\mathbb{R}^1\pi_*\mathcal{A}' = 0$.

Lemma 3.6.4. *In the situation above, the following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{Sym}(\pi_*\mathcal{V}) \otimes \pi_*\mathcal{B} \arrow[r, "\psi"] \arrow[d, "\mathrm{Id} \otimes \partial"'] & H^1(E) \arrow[d, "H^1(E \to \mathcal{O}_T)"] \\
\mathrm{Sym}(\pi_*\mathcal{V}) \otimes \mathbb{R}^1\pi_*\mathcal{V}(-\mathscr{G}) \arrow[r, "dw_T"'] & \mathcal{O}_T
\end{tikzcd} \tag{3.34}$$

Proof. First, notice that by Euler's Homogeneous Function Theorem[1], the following diagram commutes,

$$\begin{tikzcd}
\mathrm{Sym}\, V \otimes V \arrow[r, "dw"] \arrow[d, "\mathrm{m}"'] & \mathbb{C}_\chi \\
\mathrm{Sym}\, V \arrow[ur, "w"] &
\end{tikzcd} \tag{3.35}$$

where

$$\mathrm{m}((a_1 \otimes ... \otimes a_t) \otimes v) := a_1 \otimes ... \otimes a_t \otimes v.$$

[1]For a homogeneous function f of degree $t+1$, $(t+1)f = \sum x_i \partial_i f$. To apply this here, we use the fact that w has no constant term (see Definition 1.4.3). In addition, we use the convention described in Equation (3.28) which ensures the proper scaling for the above commutativity.

Recall that, working over T, we define $\operatorname{Sym}^d \mathfrak{C} := [\overbrace{\mathfrak{C} \times_T \ldots \times_T \mathfrak{C}}^{d-times} / S_d]$. Define a map

$$m_d : \operatorname{Sym}^d \mathfrak{C} \times_T \mathfrak{C} \to \operatorname{Sym}^{d+1} \mathfrak{C}.$$

In what follows, consider $\mathcal{V}^{\boxtimes d+1}$ on $\operatorname{Sym}^d \mathfrak{C} \times_T \mathfrak{C}$ with the S_d-equivariant structure given by permuting the first d-terms and as a sheaf on $\operatorname{Sym}^{d+1} \mathfrak{C}$ as a S_{d+1}-equivariant sheaf by permuting all terms. Similarly, $\mathcal{V}^{\otimes d}$ may be consider as a sheaf on $\mathfrak{C}/S_d$ where the S_d-equivariant structure comes from permuting all terms. From this we get a commutative diagram of sheaves on $\operatorname{Sym}^{d+1} \mathfrak{C}$:

$$\begin{tikzcd}
(m_d)_*\mathcal{V}^{\boxtimes d} \boxtimes \mathcal{V}(-\mathscr{G}) \arrow[r] \arrow[d] & (m_d)_*((\Delta_d)_*\mathcal{V}^{\otimes d} \boxtimes \mathcal{V}(-\mathscr{G})) \arrow[d] & \\
(m_d)_*\mathcal{V}^{\boxtimes d+1} \arrow[r] \arrow[d] & (m_d)_*((\Delta_d)_*\mathcal{V}^{\otimes d} \boxtimes \mathcal{V}) \arrow[d] & \\
\mathcal{V}^{\boxtimes d+1} \arrow[r] & (\Delta_{d+1})_*\mathcal{V}^{\otimes d+1} \arrow[d] & \\
& (\Delta_{d+1})_*\omega_{\mathfrak{C}}^{\log} \arrow[r] & (\sigma_{d+1})_*\omega_{\mathfrak{C}}^{\log}
\end{tikzcd}$$

Notice that the maps from the top two terms in the diagram to the last term are zero. Hence, we may replace the rest of the diagram by kernels of the resulting 5 maps to $(\sigma_{d+1})_*\omega_{\mathfrak{C}}^{\log}$ to get a diagram

$$\begin{tikzcd}
(m_d)_*\mathcal{V}^{\boxtimes d} \boxtimes \mathcal{V}(-\mathscr{G}) \arrow[r] \arrow[d] & (m_d)_*((\Delta_d)_*\mathcal{V}^{\otimes d} \boxtimes \mathcal{V}(-\mathscr{G})) \arrow[d] \\
\ker_1 \arrow[r] \arrow[d] & \ker_2 \arrow[d] \\
\ker_3 \arrow[r] \arrow[dr, "f_d"'] & \ker_4 \arrow[d] \\
& (\Delta_{d+1})_*\omega_{\mathfrak{C}}
\end{tikzcd}$$

Notice that after applying $(\pi_d)_*$ the right hand side of the diagram corresponds to a sequence of functors applied to $w \circ m = dw$ from (3.35). Hence the map from top to bottom pushes forward to the d^{th} component of dw.

Applying $\mathbb{R}(\pi_d)_*$, therefore leads to the following commutative diagram:

$$
\begin{tikzcd}
\operatorname{Sym}\pi_*\mathcal{V}\otimes B[-1] \arrow[d] & \\
\operatorname{Sym}\pi_*\mathcal{V}\otimes\mathbb{R}\pi_*\mathcal{V}(-\mathscr{G}) \arrow[r] \arrow[d] & \pi_*\operatorname{Sym}\mathcal{V}\otimes\mathbb{R}\pi_*\mathcal{V}(-\mathscr{G}) \arrow[d] \\
C(\operatorname{Sym}\mathbb{R}\pi_*\mathcal{V}\otimes\mathbb{R}\pi_*\mathcal{V}) \arrow[r] \arrow[d] & C(\mathbb{R}\pi_*(\operatorname{Sym}\mathcal{V}\otimes\mathcal{V})) \arrow[d] \\
C(\operatorname{Sym}\mathbb{R}\pi_*\mathcal{V}) \arrow[r] \arrow[dr] \arrow[ddr, "{(\alpha^\vee,\mathrm{sum})}"'] & C(\mathbb{R}\pi_*(\operatorname{Sym}\mathcal{V})) \arrow[d, "\sum_{d=0}^{d_0}\mathbb{R}(\pi_d)_*f_d"'] \\
& \mathbb{R}\pi_*\omega_{\mathfrak{C}} \arrow[d, "\mathrm{H}^1"] \\
& \mathcal{O}_T[-1]
\end{tikzcd}
$$

where d_0 is the top degree of w and for each complex D, we denote by $C(D)$ the mapping cone of the map $D \to \mathcal{O}_T^r$. Applying H^1 to the above, the left path becomes the top of the diagram in the statement of the lemma. The right path becomes the bottom.

□

Lemma 3.6.5. *Let $E_\bullet$ and $F_\bullet$ be quasi-isomorphic 2-term complexes of vector bundles over a smooth Deligne–Mumford stack with quasi-projective coarse moduli space. There exists a roof diagram*

$$
\begin{tikzcd}
& \bar{G}_1 \arrow[r, "d_{\bar{G}}"] \arrow[dl, two heads] \arrow[dr, two heads] & \bar{G}_0 \arrow[dl, two heads] \arrow[dr, two heads] & \\
E_1 \arrow[r, "d_E"] & E_0 & F_1 \arrow[r, "d_F"] & F_0
\end{tikzcd}
\tag{3.36}
$$

realizing the quasi-isomorphism at the cochain level, where $\bar{G}_1$ and $\bar{G}_0$ are vector bundles and all diagonal arrows are surjective.

Proof. First, there exists a roof diagram

$$E_\bullet \leftarrow H_\bullet \rightarrow F_\bullet,$$

such that both maps of complexes are quasi-isomorphisms. Replacing $H_\bullet$ with

$$K_\bullet = [K_1 = H_1/\operatorname{im}(d_H^{-1}) \to K_0 = \ker(d_H^1)]$$

yields a quasi-isomorphic complex which still maps to $E_\bullet$ and $F_\bullet$. There exists a surjection onto K_0 by a locally free coherent sheaf G_0.

Now, the mapping cone of the morphism of complexes $[0 \to G_0] \to [K_1 \to K_0]$ is the complex

$$[G_0 \oplus K_1 \to K_0].$$

Since the map from $G_0 \oplus K_1$ to K_0 is surjective, this complex has a single cohomology group which we denote by G_1. This yields a complex

$$G_\bullet := [G_1 \to G_0]$$

and a quasi-isomorphism

$$G_\bullet \to K_\bullet.$$

Since the map $K_\bullet \to E_\bullet$ is also a quasi-isomorphism, the mapping cone of the morphism $[0 \to G_0] \to [E_1 \to E_0]$ also has a single cohomology group, namely G_1. Hence, G_1 can also be realized as the kernel of the surjective morphism of vector bundles

$$G_0 \oplus E_1 \to E_0.$$

It follows that G_1 is also a vector bundle.

This realizes the isomorphism $E_\bullet$ with $F_\bullet$ in the derived category, by a roof diagram

$$E_\bullet \xleftarrow{e_\bullet} G_\bullet \xrightarrow{f_\bullet} F_\bullet \tag{3.37}$$

where $G_\bullet$ is a two-term complex of vector bundles. It is left to show that we can modify this diagram so that the morphisms of cochain complexes consist of surjective maps from G_i.

Let

$$\bar{G}_\bullet := G_1 \oplus E_1 \oplus F_1 \xrightarrow{d_{\bar{G}}=(d_G,\mathrm{id}_{E_1},\mathrm{id}_{F_1})} G_0 \oplus E_1 \oplus F_1.$$

Then we can replace (3.37) with

$$E_\bullet \xleftarrow{\bar{e}_\bullet} \bar{G}_\bullet \xrightarrow{\bar{f}_\bullet} F_\bullet,$$

where $\bar{e}_i = e_i \circ \pi_0$ is the composition of e_i with the projection onto G_i, and $\bar{f}_i$ is defined similarly. Finally, if we define

$$\hat{e}_1 = (e_1 + \mathrm{id}_{E_1}) \circ \pi_{12} \text{ and } \hat{e}_0 = (e_0 + d_E) \circ \pi_{12},$$

where π_{12} is projection onto $G_i \oplus E_1$, the map $\hat{e}_\bullet : \bar{G}_\bullet \to E_\bullet$ is homotopic to $\bar{e}_\bullet$. It is clear that $\hat{e}_1$ is surjective. Since $e_\bullet$ maps the cokernel of $G_\bullet$ onto the cokernel of $E_\bullet$, every element of E_0 must be contained in $\mathrm{im}(d_E) + \mathrm{im}(e_0)$, so $\hat{e}_0$ is surjective as well.

The analogous construction of $\hat{f}_\bullet$ gives the desired diagram (3.36).

□

With the above lemmas proven, we have concluded the proof of Proposition 3.6.1.

3.7. ***R*-charge equivariance.** Recall that $V = \oplus_{\eta \in \widehat{\mathbb{C}^*_R}} V_\eta$ comes with a *lower* grading, induced by the R-charge action. Since the actions of $\mathbb{C}^*_R$ and G commute by assumption, each V_η is a G-invariant subspace. Setting $\mathcal{V}_\eta := P \times_\Gamma V_\eta$, it follows that the vector bundle $\mathcal{V}$ on $\mathfrak{C}$ has the induced grading $\mathcal{V} = \oplus_{\eta \in \widehat{\mathbb{C}^*_R}} \mathcal{V}_\eta$, and similarly $\mathbb{R}\pi_*\mathcal{V} = \oplus_{\eta \in \widehat{\mathbb{C}^*_R}} \mathbb{R}\pi_*\mathcal{V}_\eta$.

We refine the discussion of admissible resolutions by imposing that the lower grading is respected. In other words, we work with the derived categories $\mathrm{D}([\mathfrak{C}/\mathbb{C}^*_R])$ and $\mathrm{D}([S/\mathbb{C}^*_R])$, where $\mathfrak{C}$ and S are given the trivial $\mathbb{C}^*_R$-actions. This means that our admissible resolutions $[A \xrightarrow{d} B] \cong \mathbb{R}\pi_*\mathcal{V}$ will have terms $A = \oplus_{\eta \in \widehat{\mathbb{C}^*_R}} A_\eta$ and $B = \oplus_{\eta \in \widehat{\mathbb{C}^*_R}} B_\eta$, the differential d will be $\mathbb{C}^*_R$-equivariant (i.e., of degree zero with respect to the lower grading), and each $[A_\eta \to B_\eta]$ will be a resolution of $\mathbb{R}\pi_*\mathcal{V}_\eta$. Conditions 1, 2, and 3 will now be required to hold in $\mathrm{D}([S/\mathbb{C}^*_R])$.

Hence, Condition 1 will require in addition that $\mathrm{ev}_A : A \to \pi_*(\mathcal{V}|_{\mathscr{G}})$ has degree zero with respect to the lower grading.

To formulate the equivariant Condition 2, recall first that $\eta_{\mathbf{deg}} : \mathbb{C}^*_R \to \Gamma \xrightarrow{\chi} \mathbb{C}^*$ is the character $t \mapsto t^{\mathbf{deg}}$ of $\mathbb{C}^*_R$. Therefore, as an object of $\mathrm{D}([\mathfrak{C}/\mathbb{C}^*_R])$, the line bundle $\mathcal{L}_\chi$ has $\mathbb{C}^*_R$-weight $\mathbf{deg}$. Then $\varkappa : \mathcal{L}_\chi \to \omega^{\log}_{\mathfrak{C}}$ of (3.6) can be viewed as an isomorphism in $\mathrm{D}([\mathfrak{C}/\mathbb{C}^*_R])$ by placing $\omega^{\log}_{\mathfrak{C}}$ in lower degree $\eta_{\mathbf{deg}}$ as well, i.e., by changing the target to $\omega^{\log}_{\mathfrak{C}} \otimes \mathbb{C}(\eta_{\mathbf{deg}})$. Similarly, by considering equivariant sheaf $\mathcal{O}_S(\eta_{\mathbf{deg}})$ as the target of (3.14) we get a morphism in $\mathrm{D}([S/\mathbb{C}^*_R])$. Condition 2 will now ask that α realizes this morphism at cochain level in $\mathrm{D}([S/\mathbb{C}^*_R])$.

The compatibility of Condition 3 becomes the equality of two $\mathbb{C}^*_R$-equivariant maps.

Proposition 3.5.2 (and similarly for Proposition 3.5.5) gives the existence of $\mathbb{C}^*_R$-equivariant resolutions under the same assumptions and with the same proof (we take the bundles $\mathcal{E}$ and $\mathcal{O}(1)$ in Lemma 3.5.3 to have $\mathbb{C}^*_R$-weight equal to zero). If $[A \xrightarrow{d} B]$ is a $\mathbb{C}^*_R$-equivariant admissible resolution, then $\mathrm{tot}(A)$ has the natural $\mathbb{C}^*_R$-action which is trivial on the base S and acts fiber-wise according to the decomposition $A = \oplus_{\eta\in\widehat{\mathbb{C}^*_R}} A_\eta$; the projection $p\colon\ \mathrm{tot}(A) \to S$ is equivariant.

The stacks $[V/G]$ and $I[V/G]$ have the natural residual $\mathbb{C}^*_R$-action, for which the inertia map $I[V/G] \to [V/G]$ is equivariant. On either $[V/G]$, or on its inertia stack, the potential w is naturally a section of $\mathcal{O}(\eta_{\mathbf{deg}}) := \mathcal{O} \otimes \mathbb{C}(\eta_{\mathbf{deg}})$.

In addition, $\mathrm{tot}(\pi_*(\mathcal{V}|_{\mathscr{G}_i})$ has the (fiber-wise) $\mathbb{C}^*_R$-action coming from the decomposition $\mathcal{V} = \oplus_{\eta\in\widehat{\mathbb{C}^*_R}} \mathcal{V}_\eta$. By its construction, the evaluation map

$$\mathrm{ev}^i_{\mathcal{V}} : \mathrm{tot}(\pi_*(\mathcal{V}|_{\mathscr{G}_i}) \to I[V/G]$$

of (3.5) is $\mathbb{C}^*_R$-equivariant. It follows that the composition

$$\mathrm{ev}^i\colon\ \mathrm{tot}(A) \xrightarrow{\mathrm{ev}^i_A} \mathrm{tot}(\pi_*(\mathcal{V}|_{\mathscr{G}_i}) \xrightarrow{\mathrm{ev}^i_{\mathcal{V}}} I[V/G]$$

is $\mathbb{C}^*_R$-equivariant. We also have the induced $\mathbb{C}^*_R$-equivariant total evaluation map

$$\mathrm{ev}\colon\ \mathrm{tot}(A) \to (I[V/G])^r,$$

where the target has the diagonal action of $\mathbb{C}^*_R$. By a slight abuse, from now on we will not distinguish notationally the $\mathbb{C}^*_R$-equivariant maps between spaces with action and the induced maps between the respective stack quotients by $\mathbb{C}^*_R$, i.e., we will also write

$$\mathrm{ev}\colon [\mathrm{tot}(A)/\mathbb{C}^*_R] \to [(I[V/G])^r/\mathbb{C}^*_R]. \tag{3.38}$$

When such ambiguity occurs, the context should make clear which map is meant.

The $\mathbb{C}^*_R$-equivariant admissible resolution $[A \xrightarrow{d} B]$ gives a $\mathbb{C}^*_R$-invariant section $\beta\colon \mathcal{O}_{\mathrm{tot}(A)} \to p^*B$ and a $\mathbb{C}^*_R$-invariant cosection $\alpha\colon p^*B \otimes \mathcal{O}_{\mathrm{tot}(A)}(\eta^{-1}_{\mathbf{deg}}) \to \mathcal{O}_{\mathrm{tot}(A)}$, satisfying

$$\alpha^\vee \circ (\beta \otimes \mathrm{id}_{\mathcal{O}_{\mathrm{tot}(A)}(\eta^{-1}_{\mathbf{deg}})}) = (\sum_{i=1}^r \mathrm{ev}^i)^* w \in H^0(\mathrm{tot}(A), \mathcal{O}_{\mathrm{tot}(A)}(\eta_{\mathbf{deg}})).$$

This yields a PV factorization

$$\{-\alpha,\beta\}\in \mathrm{D}([\mathrm{tot}(A)/\mathbb{C}^*_R], -(\sum_{i=1}^{r}\mathrm{ev}^i)^*w)_{[Z(dw_{\mathrm{tot}(\pi_*\mathcal{V})}(\exp\tau_{\pi_*\mathcal{V}}))/\mathbb{C}^*_R]}.$$

4. Construction of a projective embedding

In this section, we aim to construct a Deligne–Mumford stack S over the Artin stack $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma)_{\omega^{\log}_{\mathfrak{C}}}$ such that it is equipped with an admissible resolution and such that the PV factorization is supported on $LG_{g,r}(\mathcal{Z},d)=Z(\alpha,\beta)$. For the moment, we can only do it for hybrid models

$$(V=V_1\oplus V_2, G, \mathbb{C}^*_R, \theta, w),$$

(see §1.4). We recall the notations $\mathcal{X}=[V_1/\!\!/_\theta G]$ and $\mathcal{T}=[V/\!\!/_\theta G]$ for the GIT quotients stacks, and $\mathcal{Z}=Z(dw)$ for the degeneracy locus. We emphasize that, unless G is finite,

$$\mathcal{X}\neq[V_1/G]\quad\text{and}\quad\mathcal{T}\neq[V/G].$$

Moreover, we recall that the projection map $\mathcal{T}\xrightarrow{q}\mathcal{X}$ realizes $\mathcal{T}$ as a total space, $\mathcal{T}=\mathrm{tot}\,\mathcal{E}$, where $\mathcal{E}$ is a vector bundle on $\mathcal{X}$ with fiber V_2.

Subject to an additional technical condition made explicit in §4.1 below, we construct the moduli space $\square=\square_{g,r,d}$ described in §1.5. The general machinery of admissible resolutions and PV factorizations from §3 will then produce a fundamental factorization

$$K=K_{g,r,d}\in \mathrm{D}([\square/\mathbb{C}^*_R], -\mathrm{ev}^*\boxplus_{i=1}^{r}w),$$

which we shall use to define the CohFT of the GLSM hybrid model.

4.1. Convexity.

Given a hybrid GLSM $(V=V_1\oplus V_2, G, \mathbb{C}^*_R, \theta, w)$, note that by definition $\langle \mathrm{J}\rangle$ acts trivially on V_1, thus

$$G_1:=G/\langle \mathrm{J}\rangle=\Gamma/\mathbb{C}^*_R$$

acts on V_1. Introduce the rigidification of $\mathcal{X}$:

$$\mathcal{X}^{\mathrm{rig}}:=[V_1/\!\!/_\theta G_1].$$

We assume here that θ is in the image of $\widehat{G_1}\to\widehat{G}$, which can always be arranged after replacing θ by an appropriate power.

Definition 4.1.1. *Consider a Deligne–Mumford stack $\mathcal{M}$ over BG_1 and assume that $\mathcal{M}\to BG_1$ is representable and smooth. The stack $\mathcal{M}$ is called* convex over *BG_1 if for any representable morphism $f:\mathcal{C}\to\mathcal{M}$ from a genus zero orbi-curve (with any number of markings) $\mathcal{C}$ over* $\mathrm{Spec}(\mathbb{C})$ *we have*

$$H^1(\mathcal{C}, f^*T_{\mathcal{M}/BG_1})=0.$$

Remark 4.1.2.
(a) Upper semi-continuity: the dimension of the vector space $H^1(\mathcal{C}, f^*T_{\mathcal{M}/BG_1})$ is an upper semi-continuous function of the point $[f:\mathcal{C}\to\mathcal{M}]$ in the moduli space of stable maps.
(b) $\mathcal{M}$ is a smooth variety: the usual definition of convexity (over $\mathrm{Spec}(\mathbb{C})$) for a smooth variety X is when for any morphism $f:\mathcal{C}\to X$ from a genus zero curve $\mathcal{C}$, we have $H^1(\mathcal{C}, f^*T_X)=0$. If a space $\mathcal{M}$ as in Definition 4.1.1 is a smooth variety and if the group G_1 is abelian, then convexity over BG_1 is

equivalent to usual convexity, as can be seen from the distinguished triangle of tangent complexes for the morphism $\mathcal{M} \to BG_1$.

(c) G_1 is abelian: in that case, a Deligne–Mumford stack $\mathcal{M}$ is convex over BG_1 if and only if it is convex over $\mathrm{Spec}(\mathbb{C})$. The reason is as follows. On $\mathcal{M}$, there is an exact sequence of locally free sheaves: $0 \to \mathcal{O}_{\mathcal{M}}^{\mathrm{rank}\, G} \to T_{\mathcal{M}/BG} \to T_{\mathcal{M}/\mathrm{Spec}(\mathbb{C})} \to 0$. Hence, it is enough to show that $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) = 0$ for a genus zero orbicurve. It is obvious since $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) = H^1(\underline{\mathcal{C}}, \mathcal{O}_{\underline{\mathcal{C}}}) = 0$, where $\underline{\mathcal{C}}$ denotes the coarse moduli of $\mathcal{C}$.

(d) G_1 is non-abelian: convexity over BG_1 is a priori a stronger condition than convexity over $\mathrm{Spec}(\mathbb{C})$, though in familiar examples of GIT quotients of the form $[V_1/\!\!/_\theta G_1]$, such as Grassmannians, both conditions hold.

(e) Orbifold case: to prove convexity in the presence of orbifold structure on $\mathcal{M}$, it is not sufficient to check the condition from Definition 4.1.1 for maps from orbi-curves with at most two markings. For instance, consider a smooth projective elliptic curve M with a nontrivial μ_2-action. Then the stack quotient $\mathcal{M}$ be $[M/\mu_2]$ has 4 orbifold $B\mu_2$ points. If $\mathcal{C}$ is an irreducible orbicurve with at most two markings then there is no nontrival representable map from $\mathcal{C}$ to $\mathcal{M}$. This can be seen by observation that the induced map $\mathcal{C} \times_{\mathcal{M}} M \to M$ must be trivial since $\mathcal{C} \times_{\mathcal{M}} M$ is $\mathbb{P}^1$. Hence, there is no genus zero stable map with at most two markings to $\mathcal{M}$, so that the condition in Definition 4.1.1 is satisfied up to two markings. However, $\mathcal{M}$ is not convex since the identity map $\mathrm{id}\colon \mathcal{C} := \mathcal{M} \to \mathcal{M}$, with the four orbifold markings, has nonvanishing $H^1(\mathcal{C}, \mathrm{id}^* T_{\mathcal{M}/B\mu_2})$.

(f) For every closed point $B\mu_m$ of a Deligne–Mumford stack $\mathcal{M}$ and for every k with $k|m$, suppose that there is a genus zero stable map with one marking $B\mu_k$ to $\mathcal{M}$. In that case, to prove the convexity of $\mathcal{M}$ over BG_1, it is enough to check the condition in Definition 4.1.1 up to two markings. This can be seen by considering comb-like genus zero curves with orbifold nodal points.

For the remainder of the paper, we assume that $\mathcal{X}^{\mathrm{rig}}$ is convex over BG_1. Explicitly, this means that for any *genus zero stable map* $f : \mathcal{C} \to \mathcal{X}^{\mathrm{rig}}$, with any number of markings and with associated principal G_1-bundle P_1, we have the vanishing $H^1(\mathcal{C}, P_1 \times_{G_1} V_1) = 0$.

4.2. **Quasi-projective embeddings.** Fix g and r in the stable range, i.e., satisfying $2g - 2 + r > 0$. We will denote by $\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d)$ the moduli stack of degree d stable maps from families of r-pointed, genus g twisted curves (Definition 3.1.1) to the Deligne–Mumford stack $\mathcal{X}^{\mathrm{rig}}$. Following the notation in [AGV08], we use $\overline{\mathcal{K}}$ to emphasize that the gerbe markings may not have sections. The corresponding moduli stack in which the gerbe markings do have sections will be denoted $\overline{\mathcal{M}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d)$. Note that when $\mathcal{X}^{\mathrm{rig}}$ is a smooth variety, these two moduli stacks are identical.

In this subsection, following an idea in [CK16], we construct a closed immersion of $\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d)$ into a smooth Deligne–Mumford stack which, in turn, is an open substack of a stack satisfying Assumption ($\star$) of §3.5.

It will be convenient to have the following definition.

Definition 4.2.1. *Recall the definitions of §3.1.1. Given a reductive algebraic group H and a choice of character $\theta \in \widehat{H} = \mathrm{Hom}(H, \mathbb{Q})$, let $\mathfrak{M}^{\mathrm{tw}}_{g,r}(BH, d)^\circ$ denote* *the open substack of $\mathfrak{M}^{\mathrm{tw}}_{g,r}(BH,d)$ where $\omega^{\log}_{\mathfrak{C}}\otimes\mathcal{L}_\theta^{\otimes M}$ is ample for M sufficiently large. Similarly, define $\mathfrak{M}^{\mathrm{orb/tw}}_{g,r}(B\Gamma)^\circ_{\omega^{\log}_{\mathfrak{C}}}$ to be the open subset of $\mathfrak{M}^{\mathrm{orb/tw}}_{g,r}(B\Gamma)_{\omega^{\log}_{\mathfrak{C}}}$ where $\omega^{\log}_{\mathfrak{C}}\otimes\mathcal{L}_\theta^{\otimes M}$ is ample for M sufficiently large.*

Let $\mathcal{C}$ denote the universal curve over the moduli of stable curves $\overline{\mathcal{M}}_{g,r}$. Its coarse moduli space $\underline{\mathcal{C}}$ is projective over $\mathrm{Spec}(\mathbb{C})$. Choose a closed immersion $\underline{\mathcal{C}}\subseteq\mathbb{P}^{N-1}$. This in turn induces a morphism

$$v:\overline{\mathcal{M}}_{g,r}\to\overline{\mathcal{M}}_{g,r}(\mathbb{P}^{N-1},e) \tag{4.1}$$

for some degree e. Consider the relative Picard stack $\mathfrak{M}_{g,r}(B\mathbb{C}^*,e)$ parametrizing line bundles of degree e on prestable curves and let $\mathfrak{M}_{g,r}(B\mathbb{C}^*,e)^\circ$ be the open locus for which the universal bundle $\mathcal{N}$ on the universal curve $\mathfrak{C}_{B\mathbb{C}^*}\xrightarrow{\pi}\mathfrak{M}_{g,r}(B\mathbb{C}^*,e)$ is π-acyclic.

The forgetful morphism $\overline{\mathcal{M}}_{g,r}(\mathbb{P}^{N-1},e)\to\mathfrak{M}_{g,r}(B\mathbb{C}^*,e)$ induces

$$h\colon\overline{\mathcal{M}}_{g,r}\to\mathfrak{M}_{g,r}(B\mathbb{C}^*,e)$$

by precomposing with the map v of (4.1). If $\hat{h}\colon\mathcal{C}\to\mathfrak{C}_{B\mathbb{C}^*}$ is the corresponding map of universal curves, we have $\rho^*(\mathcal{O}_{\mathbb{P}^{N-1}}(1)|_{\underline{\mathcal{C}}})=\hat{h}^*\mathcal{N}$, with $\rho\colon\mathcal{C}\to\underline{\mathcal{C}}$ the coarse moduli map. We may assume the map h factors through $\mathfrak{M}_{g,r}(B\mathbb{C}^*,e)^\circ$ (if necessary, compose with a sufficiently high Veronese embedding $\mathbb{P}^{N-1}\hookrightarrow\mathbb{P}^{N'-1}$, replace $\mathbb{P}^{N-1}$ by $\mathbb{P}^{N'-1}$, and change e accordingly). This in particular implies that the image of the morphism v is contained in the open locus $\overline{\mathcal{M}}_{g,r}(\mathbb{P}^{N-1},e)^\circ$ parametrizing unobstructed stable maps to $\mathbb{P}^{N-1}$ for which maps are closed immersions of stable curves.

Next, let $\mathrm{st}\colon\mathfrak{M}^{\mathrm{tw}}_{g,r}\to\overline{\mathcal{M}}_{g,r}$ be the stabilization map. For any algebraic stack S with a morphism to $\mathfrak{M}^{\mathrm{tw}}_{g,r}$ we have the diagram

$$\begin{tikzcd}
\mathfrak{C}_S \arrow[r]\arrow[d] & \mathfrak{C} \arrow[r]\arrow[d] & \mathcal{C} \arrow[r, "\rho"]\arrow[d] & \underline{\mathcal{C}}\subseteq\mathbb{P}^{N-1} \\
S \arrow[r] & \mathfrak{M}^{\mathrm{tw}}_{g,r} \arrow[r, "\mathrm{st}"] & \overline{\mathcal{M}}_{g,r} &
\end{tikzcd} \tag{4.2}$$

The pull-back of $\mathcal{O}_{\mathbb{P}^{N-1}}(1)$ via the composition of the maps in the top row is a line bundle on the universal curve $\mathfrak{C}_S$, which we denote by $\mathcal{N}_S$.

Apply this to the forgetful map $\mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1,d)\to\mathfrak{M}^{\mathrm{tw}}_{g,r}$ to get a line bundle that we denote by $\mathcal{N}_{BG_1}$ instead of $\mathcal{N}_{\mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1,d)}$ for simplification. Let $\mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1,d)^{\circledcirc}$ be the open substack of $\mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1,d)^\circ$ for which

$$\mathbb{R}^1\pi_*(\mathcal{V}_1(-\mathscr{G})\otimes\mathcal{N}_{BG_1})=0, \tag{4.3}$$

where $\mathcal{V}_1:=\mathcal{P}_1\times_{G_1}V_1$ is defined by the universal principal G_1-bundle $\mathcal{P}_1$. [2]

Similarly, the composition $\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}},d)\to\mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1,d)\to\mathfrak{M}^{\mathrm{tw}}_{g,r}$ gives a line bundle $\mathcal{N}_{\mathcal{X}^{\mathrm{rig}}}$ on the universal curve $\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}}}$ of $\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}},d)$, compatible with $\mathcal{N}_{BG_1}$. Possibly after replacing $\mathcal{N}_{\mathcal{X}^{\mathrm{rig}}}$ by $\mathcal{N}_{\mathcal{X}^{\mathrm{rig}}}^{\otimes m}$ for some large enough m depending on g,

[2] In fact the constructions of this section work just as well over $\mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1,d)^\circ$ as over $\mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1,d)^{\circledcirc}$. However the vanishing condition (4.3) may have useful future applications which is why we choose to work over $\mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1,d)^{\circledcirc}$.

d, r (using the Veronese embedding, see above), we may assume that the forgetful map

$$\begin{tikzcd} \overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d) \arrow[r] \arrow[dr, dashed] & \mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1, d) \\ & \mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1, d)^{\circledcirc} \arrow[u, hook] \end{tikzcd} \tag{4.4}$$

factors through $\mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1, d)^{\circledcirc}$.

Remark 4.2.2. It is important to notice that for the factorization of the diagram (4.4) to hold, the convexity over BG_1 of $\mathcal{X}^{\mathrm{rig}}$ is needed, as twisting by $\mathcal{N}_{\mathcal{X}^{\mathrm{rig}}}$ cannot be used to control the required H^1-vanishing on rational tails and rational bridges of the domain curves.

Let $\mathbb{C}^*$ act on $\mathbb{C}^N$ diagonally, where we recall that the integer N is fixed by the closed immersion $\underline{\mathcal{C}} \subseteq \mathbb{P}^{N-1}$. Consider $V_1 \otimes \mathbb{C}^N$ with the G_1-action on the first factor and with the $\mathbb{C}^*$-action on the second second factor. Define a quotient stack $\mathcal{Y}$ as follows:

$$\begin{aligned} \mathcal{Y} := & \left[\left((\mathbb{C}^N \setminus \{0\}) \times (V_1 \otimes \mathbb{C}^N)^{ss}(\theta)\right) / (\mathbb{C}^* \times G_1)\right] \\ = & \left(\mathbb{C}^N \setminus \{0\}\right) \times_{\mathbb{C}^*} [(V_1 \otimes \mathbb{C}^N)^{ss}(\theta)/G_1]. \end{aligned} \tag{4.5}$$

By the Hilbert-Mumford criterion, the point $(v_1, ..., v_N) \in V_1 \otimes \mathbb{C}^N$ is θ-semistable if and only if some v_i is in $V_1^{ss}(\theta)$. It follows there are no strictly semistable points in $V_1 \otimes \mathbb{C}^N$ and hence $\mathcal{Y}$ is a separated Deligne–Mumford stack. It is clear from its construction that $\mathcal{Y}$ is a fiber bundle over the projective space $\mathbb{P}^{N-1}$, with fiber $[(V_1 \otimes \mathbb{C}^N)^{ss}(\theta)/G_1]$. This fibration is not trivial, as it is twisted by the transition functions of $\mathcal{O}_{\mathbb{P}^{N-1}}(1)$. Moreover, the coarse space map $\underline{\mathcal{Y}} \to \mathbb{P}^{N-1}$ is a projective morphism, since $(\mathrm{Sym}(V_1^\vee))^{G_1} = \mathbb{C}$ by the definition of hybrid GLSM. In particular, the coarse space $\underline{\mathcal{Y}}$ is projective over $\mathrm{Spec}(\mathbb{C})$.

Considering $d' := (d, e)$ as an element in $\widehat{G_1 \times \mathbb{C}^*}$, we have the moduli stack $\overline{\mathcal{K}}_{g,r}(\mathcal{Y}, d')$. The projection $\mathcal{Y} \to \mathbb{P}^{N-1}$ induces a map

$$\mathrm{pr}\colon \overline{\mathcal{K}}_{g,r}(\mathcal{Y}, d') \to \overline{\mathcal{M}}_{g,r}(\mathbb{P}^{N-1}, e).$$

On the other hand via the forgetful map $\mathrm{fgt}\colon \overline{\mathcal{K}}_{g,r}(\mathcal{Y}, d') \to \mathfrak{M}^{\mathrm{tw}}_{g,r}$, we also have

$$v \circ \mathrm{st} \circ \mathrm{fgt}\colon \overline{\mathcal{K}}_{g,r}(\mathcal{Y}, d') \to \overline{\mathcal{M}}_{g,r}(\mathbb{P}^{N-1}, e).$$

Define $\overline{\mathcal{K}}^{eq}_{g,r}(\mathcal{Y}, d')$ as the fiber product

$$\begin{tikzcd} \overline{\mathcal{K}}^{eq}_{g,r}(\mathcal{Y}, d') \arrow[r, hook] \arrow[d, "\mathrm{fgt}"'] & \overline{\mathcal{K}}_{g,r}(\mathcal{Y}, d') \arrow[d, "(\mathrm{fgt},\mathrm{pr})"] \\ \mathfrak{M}^{\mathrm{tw}}_{g,r} \arrow[r, hook, "(\mathrm{id}, v\circ\mathrm{st})"'] & \mathfrak{M}^{\mathrm{tw}}_{g,r} \times \overline{\mathcal{M}}_{g,r}(\mathbb{P}^{N-1}, e). \end{tikzcd} \tag{4.6}$$

Since $\overline{\mathcal{M}}_{g,r}(\mathbb{P}^{N-1})^\circ$ consists of closed immersions, the image of v is contained in the separated *scheme* $\overline{\mathcal{M}}_{g,r}(\mathbb{P}^{N-1})^\circ$, therefore the "graph of $v \circ \mathrm{st}$" morphism $(\mathrm{id}, v \circ \mathrm{st})$ is a representable closed immersion and thus $\overline{\mathcal{K}}^{eq}_{g,r}(\mathcal{Y}, d')$ is a closed substack of $\overline{\mathcal{K}}_{g,r}(\mathcal{Y}, d')$. More informally, $\overline{\mathcal{K}}^{eq}_{g,r}(\mathcal{Y}, d')$ is the closed substack of $\overline{\mathcal{K}}_{g,r}(\mathcal{Y}, d')$ on which the equality $v \circ \mathrm{st} \circ \mathrm{fgt} = \mathrm{pr}$ holds.

Remark 4.2.3. It is straightforward to check that for a scheme S, the S-points of $\overline{\mathcal{K}}^{eq}_{g,r}(\mathcal{Y},d')$ are given by families

$$((\mathfrak{C}_S \to S, \mathscr{G}_1, \dots, \mathscr{G}_r, P_1, \{u_1, \dots u_N\}), \tag{4.7}$$

where $(\mathfrak{C}_S \to S, \mathscr{G}_1, \dots, \mathscr{G}_r)$ is a twisted r-pointed curve of genus g over S, P_1 is a principal G_1 bundle of degree d on $\mathfrak{C}_S$, and u_i are sections of $\mathcal{V}_1 \otimes \mathcal{N}_S$, satisfying an appropriate stability condition. Here $\mathcal{V}_1 = P_1 \times_{G_1} V_1$, and $\mathcal{N}_S$ is the line bundle defined by the map $S \to \mathfrak{M}^{\mathrm{tw}}_{g,r}$ via the diagram (4.2). The stability condition requires first that the map to $V_1 \otimes \mathbb{C}^N$ given by the sections lands in $(V_1 \otimes \mathbb{C}^N)^{ss}(\theta)$, and second that $\omega^{\log}_{\mathfrak{C}} \otimes \mathcal{L}^M_\theta$ is relatively ample for large enough M.

In fact, the stack $\overline{\mathcal{K}}^{eq}_{g,r}(\mathcal{Y},d')$ could have been *defined* directly, without reference to $\mathcal{Y}$, as the moduli stack parametrizing the families (4.7). It is clear that this moduli definition gives an algebraic stack. In fact, it gives an open substack in the cone $\operatorname{tot}\left(\pi_*(\mathcal{V}_1 \otimes \mathcal{N}^{\oplus N}_{BG_1})\right)$ over $\mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1,d)^\circ$. Stability implies it is a Deligne–Mumford stack. However, its concrete realization given by (4.6) also implies immediately the following additional properties: the Deligne–Mumford stack $\overline{\mathcal{K}}^{eq}_{g,r}(\mathcal{Y},d')$ is a global quotient by a linear algebraic group, and it has projective coarse moduli. Indeed, the ambient stack $\overline{\mathcal{K}}_{g,r}(\mathcal{Y},d')$ has these properties by [AGOT07, Theorem 1.0.2 & §3.2] and [AV02, Theorem 1.4.1].

Define $U_{\mathcal{Y}}$ by the fiber square

$$\begin{tikzcd} U_{\mathcal{Y}} \arrow[r] \arrow[d] & \overline{\mathcal{K}}^{eq}_{g,r}(\mathcal{Y},d') \arrow[d] \\ \mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1,d)^{\circledcirc} \arrow[r] & \mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1,d)^\circ, \end{tikzcd} \tag{4.8}$$

so that $U_{\mathcal{Y}}$ is an open substack of $\overline{\mathcal{K}}^{eq}_{g,r}(\mathcal{Y},d')$.

Let $\mathcal{N}^{eq}$ be the line bundle on the universal curve over $\overline{\mathcal{K}}^{eq}_{g,r}(\mathcal{Y},d')$ induced by the map fgt via (4.2), and note that it coincides with the pull-back of $\mathcal{N}_{BG_1}$. By its moduli description in Remark 4.2.3, the stack $\overline{\mathcal{K}}^{eq}_{g,r}(\mathcal{Y},d')$ has a perfect obstruction theory relative to $\mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1,d)$, given by

$$\mathbb{R}\pi_*((\mathcal{V}_1 \otimes \mathcal{N}^{eq})^{\oplus N}).$$

Hence the obstruction sheaf is isomorphic to $\mathbb{R}^1\pi_*((\mathcal{V}_1 \otimes \mathcal{N}^{eq})^{\oplus N})$, which is a quotient of $\mathbb{R}^1\pi_*((\mathcal{V}_1(-\mathscr{G}) \otimes \mathcal{N}^{eq})^{\oplus N})$. By (4.3), the obstruction sheaf vanishes when restricted to $U_{\mathcal{Y}}$. This implies that $U_{\mathcal{Y}}$ is smooth over $\operatorname{Spec}(\mathbb{C})$.

Lemma 4.2.4. *Fix (g,r) with $2g-2+r>0$.*

(*a*) *There is a two-term locally free resolution*

$$[A_1 \xrightarrow{d_1} B_1] \cong \mathbb{R}\pi_*\mathcal{V}_1$$

on $\mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1,d)^{\circledcirc}$, and a natural open immersion $U_{\mathcal{Y}} \subseteq \operatorname{tot} A_1$, compatible with the maps to $\mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1,d)^{\circledcirc}$.

(*b*) *Let $p_1\colon \operatorname{tot} A_1 \to \mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1,d)^{\circledcirc}$ denote the projection and let β_1 be the section of $p_1^*B_1|_{U_{\mathcal{Y}}}$ induced from d_1, with zero locus $Z(\beta_1) \subseteq U_{\mathcal{Y}}$. Assume that $\mathcal{X}^{\mathrm{rig}}$ is convex. Then there is a canonical identification $\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}},d) \cong Z(\beta_1)$, so that* *we have a closed immersion making a commuting diagram*

$$\begin{tikzcd}
\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d) \arrow[r, hook] \arrow[dr] & U_{\mathcal{Y}} \arrow[d] \\
& \mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1, d)^{\circledcirc}.
\end{tikzcd}$$

Proof. (*a*) Consider the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{N-1}} \to \mathcal{O}_{\mathbb{P}^{N-1}}(1)^{\oplus N} \to T_{\mathbb{P}^{N-1}} \to 0$$

on $\mathbb{P}^{N-1}$. Pulling it back via the composition $\mathfrak{C}_{BG_1} \to \underline{\mathcal{C}} \to \mathbb{P}^{N-1}$ and tensoring with $\mathcal{V}_1$, we obtain a short exact sequence

$$0 \to \mathcal{V}_1 \to \mathcal{V}_1 \otimes \mathcal{N}_{BG_1}^{\oplus N} \xrightarrow{q} \mathcal{Q} \to 0 \tag{4.9}$$

where $\mathcal{Q}$ is the cokernel. Pushing forward via π_* and using the vanishing (4.3) gives the long exact sequence

$$0 \to \pi_*(\mathcal{V}_1) \to \pi_*(\mathcal{V}_1 \otimes \mathcal{N}_{BG_1}^{\oplus N}) \xrightarrow{\pi_* q} \pi_*(\mathcal{Q}) \to \mathbb{R}^1\pi_*(\mathcal{V}_1) \to 0, \tag{4.10}$$

on $\mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1, d)^{\circledcirc}$, with the middle two terms locally free. Hence

$$[A_1 \to B_1] := [\pi_*(\mathcal{V}_1 \otimes \mathcal{N}_{BG_1}^{\oplus N}) \to \pi_*(\mathcal{Q})]$$

gives the required complex of vector bundles on the stack $\mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1, d)^{\circledcirc}$. Clearly $U_{\mathcal{Y}}$ is the open locus in tot A_1 obtained by imposing the condition that the sections in $A_1 = \pi_*(\mathcal{V}_1 \otimes \mathcal{N}_{BG_1}^{\oplus N})$ give a map landing in $(V_1 \otimes \mathbb{C}^N)^{ss}(\theta)$.

(*b*) This follows from (4.10) and (4.4). $\square$

4.3. **Properties of moduli of LG maps.** In this subsection we show the moduli stacks of Landau–Ginzburg maps to the "base" $\mathcal{X} = [V_1 /\!\!/_\theta G]$ of a hybrid model $(V = V_1 \oplus V_2, G, \mathbb{C}^*_R, \theta, w)$ are global quotient stacks with projective coarse moduli. These properties will follow from the corresponding ones for the related spaces of stable maps to $\mathcal{X}^{\mathrm{rig}}$ discussed in the previous subsection.

Recall from §1 that the group Γ surjects onto $G_1 \times \mathbb{C}^*$ via the map $g \cdot \lambda \mapsto ([g], \lambda^{\mathbf{deg}})$, where $G_1 := G/\langle J \rangle$. It yields the exact sequence

$$1 \to \langle J \rangle \to \Gamma \to G_1 \times \mathbb{C}^* \to 1.$$

Borrowing an idea from [AJ03, §1.5], we consider the fiber diagram:

$$\begin{tikzcd}
\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma} \arrow[r] \arrow[d] & B\Gamma \arrow[d] \\
\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}}} \arrow[r, "{[\mathcal{P}] \times \omega^{\log}_{\mathfrak{C}}}"] \arrow[d] & BG_1 \times B\mathbb{C}^* \\
\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d) &
\end{tikzcd} \tag{4.11}$$

where $\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}}}$ is the universal curve, $[\mathcal{P}]$ is the universal principal G_1-bundle, and $\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma}$ is defined as the fiber product. The map $B\Gamma \to BG_1 \times B\mathbb{C}^*$ is an étale gerbe, therefore the map $\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma} \to \mathfrak{C}_{\mathcal{X}^{\mathrm{rig}}}$ is also an étale gerbe.

For each connected component of the space $\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d)$, let F be the class of a fiber of $\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}}} \to \overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d)$ in the homology group of $\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}}}$. Let us consider the moduli space

$$\overline{\mathcal{M}}_{g,r}\left(\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma}/\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d), F\right)$$

of balanced stable maps from orbi-curves to $\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma}$ *relative to* $\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d)$, of class F, see [AV02, §8.3].

Remark 4.3.1. In [AV02], the notation $\overline{\mathcal{K}}_{g,r}\left(\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma}/\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d), F\right)$ is used to denote the space of relative twisted stable maps. In this case, the marked point divisors on the universal curve are (nontrivial) gerbes over the moduli space. In keeping with the notation of [AGV08, §6.1.3], we use $\mathcal{M}$ to denote the space of relative stable maps *with sections at the gerbe markings* (recall part *(a)* of Definition 1.2.1). $\overline{\mathcal{M}}_{g,r}\left(\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma}/\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d), F\right)$ is a finite cover of $\overline{\mathcal{K}}_{g,r}\left(\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma}/\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d), F\right)$, and is easily constructed as in §3.1.1 and [AGV08, §6.1.3] as

$$\overline{\mathcal{M}}_{g,r}\left(\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma}/\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d), F\right) := \mathscr{G}_1 \times_{\overline{\mathcal{K}}} \mathscr{G}_2 \times_{\overline{\mathcal{K}}} \cdots \times_{\overline{\mathcal{K}}} \mathscr{G}_r, \tag{4.12}$$

where $\overline{\mathcal{K}}$ here denotes $\overline{\mathcal{K}}_{g,r}\left(\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma}/\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d), F\right)$. Note that by construction, there is a forgetful map $\overline{\mathcal{M}}_{g,r}\left(\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma}/\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d), F\right) \to \mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)$.

Recall that a map

$$S \to LG_{g,r}(\mathcal{X}, d)$$

is a family over S of prestable orbicurves with gerbe markings, sections of these gerbes, a principal Γ-bundle $\mathcal{P}$, an isomorphism $\chi_*(\mathcal{P}) \simeq \omega^{\log}_{\mathfrak{C}}$, and sections of the associated vector bundle $\mathcal{V}_1$ satisfying some stability conditions. Composing with the map $B\Gamma \to BG_1 \times B\mathbb{C}^*$, the bundle $\mathcal{P}$ induces a principal G_1-bundle $[\mathcal{P}]$ and a line bundle isomorphic to $\omega^{\log}_{\mathfrak{C}}$. Moreover, the vector bundle $\mathcal{V}_1$ is also the associated vector bundle to $[\mathcal{P}]$. Thus we get a family over S of prestable orbicurves with gerbe markings, with the bundle $[\mathcal{P}]$, and with sections of the vector bundle $\mathcal{V}_1$ satisfying the same stability conditions as before. After partially rigidifying the curve in the sense of [AV02, Proposition 9.1.1] we obtain a map:

$$S \to \overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d).$$

As a consequence, we get a morphism

$$LG_{g,r}(\mathcal{X}, d) \to \overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d) \tag{4.13}$$

Hence, the universal curve over $LG_{g,r}(\mathcal{X}, d)$ maps to the universal curve $\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}}}$ over $\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d)$. But it also maps to $B\Gamma$ via the universal bundle $\mathcal{P}$. Both $B\Gamma$ and $\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}}}$ map to $BG_1 \times B\mathbb{C}^*$. By construction of the morphism (4.13), the corresponding maps from the universal curve over $LG_{g,r}(\mathcal{X}, d)$ to $BG_1 \times B\mathbb{C}^*$ are equal, so that we get a map from the universal curve over $LG_{g,r}(\mathcal{X}, d)$ to $\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma}$. Such a map induces a morphism

$$LG_{g,r}(\mathcal{X}, d) \to \overline{\mathcal{M}}_{g,r}\left(\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma}/\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d), F\right). \tag{4.14}$$

More precisely, this map is a closed immersion and the stack $LG_{g,r}(\mathcal{X}, d)$ can be identified with the closed substack consisting of stable maps f such that, for every marking σ_i, the point $f(\sigma_i)$ is sent to the corresponding marking $\sigma_i \in \mathfrak{C}_{\mathcal{X}^{\mathrm{rig}}}$ via the map $\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma} \to \mathfrak{C}_{\mathcal{X}^{\mathrm{rig}}}$.

Proposition 4.3.2. (*a*) *The morphism $LG_{g,r}(\mathcal{X}, d) \to \overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d)$ defined in* (4.13) *is proper, quasi-finite, and of Deligne–Mumford type.*

(*b*) *The stack $LG_{g,r}(\mathcal{X}, d)$ is a global quotient stack with projective coarse moduli space.*

Proof. (a) The properties listed follow from the description (4.14) of $LG_{g,r}(\mathcal{X}, d)$ as a space of relative stable maps (see [AV02, Theorem 1.4.1 and Section 8.3]).

(b) This is due to Lemma 4.3.3 below and [AGOT07, Corollary 1.0.3]. □

Lemma 4.3.3. *The stack $\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma}$ is a global quotient by a linear algebraic group, and has projective coarse moduli space.*

Proof. First note that the coarse moduli of $\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma}$ and $\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}}}$ coincide, and that $\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}}}$ has projective coarse moduli by [AV02].

Second, $\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}}}$ is a quotient stack by [AGOT07]. Hence there is a scheme R and a finite flat surjective morphism $R \to \mathfrak{C}_{\mathcal{X}^{\mathrm{rig}}}$, by [Kre09, Proposition 5.1]. Let P be the $G_1 \times \mathbb{C}^*$-bundle on R corresponding to the map $R \to BG_1 \times B\mathbb{C}^*$. There is a natural map from the algebraic space P to the fiber product $R' := R \times_{\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}}}} \mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma}$, realizing R' as a global quotient. The scheme R is projective, since the map from R to the coarse moduli of $\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}}}$ is finite. Therefore R' has projective coarse moduli R. Using again [Kre09, Proposition 5.1], there is a scheme R'' and a finite flat surjective morphism $R'' \to R'$. The composition $R'' \to \mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma}$ is finite, flat, and surjective.

Applying [Kre09, Proposition 5.1] a third time (now in reverse direction), we conclude that $\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma}$ is a global quotient. □

Note that the above construction can be repeated over the space $\overline{\mathcal{K}}^{eq}_{g,r}(\mathcal{Y}, d)$. We have the following fiber product diagram, defining $\mathfrak{C}_{\mathcal{Y},\Gamma}$,

$$\begin{tikzcd}
 & \mathfrak{C}_{\mathcal{Y},\Gamma} \arrow[rd] \arrow[dd] & \\
\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma} \arrow[ru, hook] \arrow[rr] \arrow[dd] & & B\Gamma \arrow[dd] \\
 & \mathfrak{C}_{\mathcal{Y}} \arrow[rd] \arrow[dd] & \\
\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}}} \arrow[ru, hook] \arrow[rr] \arrow[dd] & & BG_1 \times B\mathbb{C}^* \\
 & \overline{\mathcal{K}}^{eq}_{g,r}(\mathcal{Y}, d') & \\
\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d) \arrow[ru, hook] & &
\end{tikzcd} \tag{4.15}$$

Define the space $LG^{eq}_{g,r}(\mathcal{Y}, d')$ as the closed substack of the moduli space of relative stable maps

$$\overline{\mathcal{M}}_{g,r}\left(\mathfrak{C}_{\mathcal{Y},\Gamma}/\overline{\mathcal{K}}^{eq}_{g,r}(\mathcal{Y}, d'), F\right)$$

consisting of stable maps f such that, for every marking σ_i, the point $f(\sigma_i)$ is sent to the corresponding marking $\sigma_i \in \mathfrak{C}_{\mathcal{Y}}$ via the map $\mathfrak{C}_{\mathcal{Y},\Gamma} \to \mathfrak{C}_{\mathcal{Y}}$. As a consequence, Diagram (4.15) shows that $LG_{g,r}(\mathcal{X}, d)$ embeds into the larger space $LG^{eq}_{g,r}(\mathcal{Y}, d')$ of Landau–Ginzburg maps.

Define an open substack U of $LG^{eq}_{g,r}(\mathcal{Y}, d')$ as the preimage of $U_{\mathcal{Y}}$ under the map $LG^{eq}_{g,r}(\mathcal{Y}, d') \to \overline{\mathcal{K}}^{eq}_{g,r}(\mathcal{Y}, d')$. The diagram

$$\begin{tikzcd} LG_{g,r}(\mathcal{X}, d) \arrow[r, hook] \arrow[d] & U \arrow[r, hook] \arrow[d] & LG^{eq}_{g,r}(\mathcal{Y}, d') \arrow[d] \\ \overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d) \arrow[r, hook] & U_{\mathcal{Y}} \arrow[r, hook] & \overline{\mathcal{K}}^{eq}_{g,r}(\mathcal{Y}, d') \end{tikzcd} \tag{4.16}$$

is cartesian, where the first pair of horizontal arrows are closed immersions and the second pair of horizontal arrows are open immersions. Note that U is smooth over $\mathrm{Spec}(\mathbb{C})$.

There is a natural morphism $B\Gamma \to BG_1$, induced by the quotient map $\Gamma \to G_1$. In turn, it induces the forgetful morphism $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)_{\omega^{\log}_{\mathfrak{C}}} \to \mathfrak{M}^{\mathrm{orb}}_{g,r}(BG_1, d) \to \mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1, d)$. Let $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)^{\circledcirc}_{\omega^{\log}_{\mathfrak{C}}}$ be the inverse image of $\mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1, d)^{\circledcirc}$ under the above forgetful morphism. The following theorem collects together the key points of the construction in this subsection.

Theorem 4.3.4. *Over $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)^{\circledcirc}_{\omega^{\log}_{\mathfrak{C}}}$, there exists a two-term resolution*

$$[A_1 \xrightarrow{d_1} B_1] \cong \mathbb{R}\pi_*(\mathcal{V}_1)$$

by vector bundles and an open substack $U \subseteq \mathrm{tot}(A_1)$ over $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)^{\circledcirc}_{\omega^{\log}_{\mathfrak{C}}}$ which is a smooth separated Deligne–Mumford stack such that:

(a) *$Z(\beta_1) = LG_{g,r}(\mathcal{X}, d)$, where $\beta_1 \in H^0(U, p_1^*B_1|_U)$ is the section induced from d_1. Here $p_1 : \mathrm{tot}\, A_1 \to \mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)^{\circledcirc}_{\omega^{\log}_{\mathfrak{C}}}$ is the projection.*

(b) *There exists an open immersion of U into a global quotient Deligne–Mumford stack $LG^{eq}_{g,r}(\mathcal{Y}, d')$, whose coarse moduli space is projective.*

Proof. Let $\gamma : \mathfrak{C}_{B\Gamma} \to \mathfrak{C}_{BG_1}$ denote the natural map between universal curves over the stacks $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)_{\omega^{\log}_{\mathfrak{C}}}$ and $\mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1, d)$, and consider the pullback of the sequence (4.9) via γ,

$$0 \to \mathcal{V}_1 \to \mathcal{V}_1 \otimes \widetilde{\mathcal{N}}^{\oplus N} \xrightarrow{\delta_1} \gamma^*\mathcal{Q} \to 0,$$

where $\widetilde{\mathcal{N}}$ denotes $\gamma^*\mathcal{N}_{BG_1}$ and, by abuse of notation, we are using $\mathcal{V}_1$ to denote bundles on the universal curves over both $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)_{\omega^{\log}_{\mathfrak{C}}}$ and $\mathfrak{M}^{\mathrm{tw}}_{g,r}(BG_1, d)$. Note that both $\mathcal{V}_1 \otimes \widetilde{\mathcal{N}}^{\oplus N}$ and $\gamma^*(\mathcal{Q})$ are π-acyclic over $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)^{\circ}_{\omega^{\log}_{\mathfrak{C}}}$. Hence we have the locally free resolution

$$[A_1 \xrightarrow{d_1} B_1] := [\pi_*(\mathcal{V}_1 \otimes \widetilde{\mathcal{N}}^{\oplus N}) \to \pi_*(\gamma^*\mathcal{Q})] \cong \mathbb{R}\pi_*\mathcal{V}_1 \tag{4.17}$$

on the stack $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)_{\omega^{\log}_{\mathfrak{C}}}$.

As in the proof of Lemma 4.2.4, the stack U from (4.16) is naturally an open substack of $\mathrm{tot}(A_1) \xrightarrow{p_1} \mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)^{\circledcirc}_{\omega^{\log}_{\mathfrak{C}}}$. To see that $LG_{g,r}(\mathcal{X}, d)$ is exactly the vanishing locus of β_1 in U, we note that the open condition defining U in $\mathrm{tot}(A_1)$, when restricted to the cone $\mathrm{tot}(\pi_*(\mathcal{V}_1))$, gives exactly the stability condition defining $LG_{g,r}(\mathcal{X}, d)$. This proves part (a).

For part (b), observe that the same argument as in the proof of Lemma 4.3.3 shows that $\mathfrak{C}_{\mathcal{Y},\Gamma}$ is a quotient stack with projective coarse moduli. By [AGOT07, Corollary 1.0.3], the space $LG^{eq}_{g,r}(\mathcal{Y}, d')$ is also a quotient stack with projective coarse moduli. $\square$

5. The GLSM theory for convex hybrid models

5.1. The Fundamental Factorization. By Theorem 4.3.4, the space $LG^{eq}_{g,r}(\mathcal{Y}, d')$ satisfies Assumption ($\star$) of §3.5. By Proposition 3.5.2, there is an admissible resolution (i.e. satisfying Conditions 1, 2, and 3)

$$[\widetilde{A}_1 \oplus \widetilde{A}_2 \to \widetilde{B}_1 \oplus \widetilde{B}_2] \cong \mathbb{R}\pi_*(\mathcal{V}_1 \oplus \mathcal{V}_2) \quad \text{on } \overline{U} \subseteq LG^{eq}_{g,r}(\mathcal{Y}, d')$$

with evaluation map $\mathrm{ev}^i_{\widetilde{A}}$ and cosection $\widetilde{\alpha}$. In particular, observe that we have two resolutions of $\mathbb{R}\pi_*\mathcal{V}_1$ lying over the space U.

Apply Lemma 3.6.5 to $[\widetilde{A}_1 \to \widetilde{B}_1] \cong [A_1 \to B_1]$ and let $\bar{A}_2 = \widetilde{A}_2$ and $\bar{B}_2 = \widetilde{B}_2$, to obtain the following diagram:

$$\begin{tikzcd}
 & \bar{A} \arrow[r, "\bar{d}"] \arrow[dl, two heads, "f_{\widetilde{A}}"'] \arrow[dr, two heads, "f_{A_1}"] & \bar{B} \arrow[dl, two heads, "f_{\widetilde{B}}"'] \arrow[dr, two heads, "f_{B_1}"] & \\
\widetilde{A} \arrow[r] & \widetilde{B} & A_1 \arrow[r, "d_1"] & B_1
\end{tikzcd} \tag{5.1}$$

where the left square is a quasi-isomorphism, the right square is a composition of the projection followed by a quasi-isomorphic cochain map $[\bar{A}_1 \to \bar{B}_1] \ \to [A_1 \to B_1]$, and the diagonal maps are all surjective. With $\mathrm{ev}^i_{\widetilde{A}}$ as in Condition 1 of Definition 3.2.1, we define $\mathrm{ev}^i_{\bar{A}}$ as the composition

$$\mathrm{ev}^i_{\bar{A}} : \bar{A} \xrightarrow{f_{\widetilde{A}}} \widetilde{A} \xrightarrow{\mathrm{ev}^i_{\widetilde{A}}} \mathcal{O}_U\,.$$

Similarly, we define $\overline{\alpha}^\vee$ as the composition

$$\overline{\alpha}^\vee : \mathrm{Sym}\,\bar{A} \otimes \bar{B} \xrightarrow{\mathrm{Sym}(f_{\widetilde{A}}) \otimes f_{\widetilde{B}}} \mathrm{Sym}\,\widetilde{A} \otimes \widetilde{B} \xrightarrow{\widetilde{\alpha}^\vee} \mathcal{O}_U\,.$$

By Lemma 3.4.3 $[\bar{A} \to \bar{B}]$ is admissible.

Let $p_{\mathrm{tot}(\bar{A})}$ denote the projection $\mathrm{tot}(\bar{A}) \to U$ and consider the section $\zeta \in \Gamma(\mathrm{tot}(\bar{A}), p^*_{\mathrm{tot}(\bar{A})}A_1)$ given by

$$\zeta = p^*_{\mathrm{tot}(\bar{A})} f_{A_1} \circ \mathrm{taut}_{\bar{A}} - p^*_{\mathrm{tot}(\bar{A})}(\mathrm{taut}_{A_1}),$$

where taut_{A_1} and $\mathrm{taut}_{\bar{A}}$ are the tautological sections of the bundles $A_1 \to U$ and $\bar{A} \to \mathrm{tot}(\bar{A})$ respectively (note that the former uses the fact that U is an open substack of $\mathrm{tot}\, A_1$).

Recalling Definition 3.3.1 there exist evaluation maps

$$\mathrm{ev}^i\colon\ \mathrm{tot}(\bar{A}) \to I[V/G] \tag{5.2}$$

We define $\mathrm{tot}(\bar{A})^\circ$ as the open substack of $\mathrm{tot}(\bar{A})$ such that ev^i maps to the semistable locus $I[V/\!\!/_\theta G]$

$$\mathrm{tot}(\bar{A})^\circ := \bigcap_{i=1}^r (\mathrm{ev}^i)^{-1} I[V/\!\!/_\theta G].$$

Definition 5.1.1. *Define the space*

$$\square = \square_{g,r,d} := \{\zeta = 0\} \subseteq \mathrm{tot}(\bar{A})^{\circ}.$$

Since the map $\bar{A} \to A_1$ is surjective, $\square$ is a smooth Deligne–Mumford stack.

Remark 5.1.2. Note that the stack $\square$ is determined by two different resolutions. First over $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)^{\circ}_{\omega^{\log}_{\mathfrak{C}}}$ we construct a resolution $[A_1 \xrightarrow{d_1} B_1]$ of $\mathbb{R}\pi_*(\mathcal{V}_1)$ by vector bundles and define an open set $U \subseteq \mathrm{tot}(A_1)$ which is a smooth separated Deligne–Mumford stack with quasiprojective coarse moduli space (see Corollary 4.3.4). Second, over U we construct a resolution $[\bar{A} \xrightarrow{\bar{d}} \bar{B}]$ of $\mathbb{R}\pi_*(\mathcal{V})$ by vector bundles. The space $\square$ is given as a subset of $\mathrm{tot}(\bar{A})$ over U. We refer to this construction involving two resolutions as the two-step procedure.

Remark 5.1.3. Due to the fact that the stack $\mathrm{tot}(\bar{A})$ lies over $U \subseteq \mathrm{tot}(A_1)$, the relative dimension of $\mathrm{tot}(\bar{A}) \to \mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)^{\circ}_{\omega^{\log}_{\mathfrak{C}}}$ is given by $\mathrm{rank}(A_1) + \mathrm{rank}(\bar{A})$. If we view the pullback of $\bar{B}$ to $\mathrm{tot}(\bar{A})$ as an obstruction bundle, the virtual dimension relative to $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)^{\circ}_{\omega^{\log}_{\mathfrak{C}}}$ is $\chi(\mathbb{R}\pi_*(\mathcal{V})) + \mathrm{rank}(A_1)$, which is dimension $\mathrm{rank}(A_1)$ more than it should be. To correct for this overcounting, we must restrict to the locus $\square$ of points $(a_1, \bar{a}) \in \mathrm{tot}(\bar{A})$ such that $f_{\widetilde{A}}(\bar{a}) = a_1$. Restricting $\bar{B}$ to this locus will then yield a stack of the correct virtual dimension.

Definition 5.1.4. *Define the vector bundle $E \to \square$ as (the pullback of) $\bar{B}$. The vector bundle E has a section $\bar{\beta}$ and a cosection $\overline{\alpha}^{\vee}$. For notational simplicity we will drop the bars and denote these simply as β and α when no confusion is likely to occur. We define*

$$K = K_{g,r,d} \in \mathrm{D}([\square_{g,r,d}/\mathbb{C}^*_R], -\mathrm{ev}^*(\boxplus_{i=1}^r w))$$

to be the Koszul factorization $\{-\alpha, \beta\}$. We refer to $\{-\alpha, \beta\}$ as the fundamental factorization.

It is straightforward to check that the above data satisfies all the properties promised in §1.5. In particular we record the following fact.

Lemma 5.1.5. *The support of $K_{g,r,d}$ is equal to $LG_{g,r}(\mathcal{Z}, d)$. In particular, it has proper support.*

Proof. Recall that by Theorem 4.3.4 (a), U is an open substack of $\mathrm{tot}\, A_1$ over $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)^{\circledcirc}_{\omega^{\log}_{\mathfrak{C}}}$ with $Z(\beta_1) = LG_{g,r}(\mathcal{X}, d)$. In particular, we may view $\mathrm{tot}\, \bar{A}$ over U as an open substack of $\mathrm{tot}\, A_1 \times_{\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)^{\circledcirc}_{\omega^{\log}_{\mathfrak{C}}}} \mathrm{tot}\, \bar{A}$. Now, by Lemma 3.6.2, the zero locus $Z(\beta)$ in $\mathrm{tot}(\bar{A})$ is isomorphic to $\mathrm{tot}\, \pi_*\mathcal{V}$ over U. Hence, we may view $Z(\beta)$ as an open subset of $\mathrm{tot}\, A_1 \times_{\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)^{\circledcirc}_{\omega^{\log}_{\mathfrak{C}}}} \mathrm{tot}\, \pi_*\mathcal{V}$.

Recall that

$$\zeta = p^*_{\mathrm{tot}(\bar{A})} f_{A_1} \circ \mathrm{taut}_{\bar{A}} - p^*_{\mathrm{tot}(\bar{A})}(\mathrm{taut}_{A_1}).$$

Since $[\bar{A}_1 \to \bar{B}_1] \to [A_1 \to B_1]$ is a quasi-isomorphism, we have a commutative diagram

$$\begin{array}{ccc} \pi_*\mathcal{V}_1 & \longrightarrow & \bar{A} \\ \big\downarrow{\scriptstyle \mathrm{Id}} & & \big\downarrow{\scriptstyle f_{A_1}} \\ \pi_*\mathcal{V}_1 & \longrightarrow & A_1. \end{array}$$

It follows that the restriction

$$\zeta|_{\operatorname{tot} A_1 \times_{\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)^{\circledcirc}_{\omega^{\log}_{\mathfrak{C}}}} \operatorname{tot} \pi_*\mathcal{V}}$$

defines an open subset of

$$\Delta \times_{\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)^{\circledcirc}_{\omega^{\log}_{\mathfrak{C}}}} \operatorname{tot} \pi_*\mathcal{V}_2$$

where Δ is diagonal map of $\operatorname{tot} \pi_*\mathcal{V}_1$ over $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)^{\circledcirc}_{\omega^{\log}_{\mathfrak{C}}}$. This is isomorphic to $\operatorname{tot} \pi_*\mathcal{V}$. By definition of U, the intersection $(\Delta \times_{\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)^{\circledcirc}_{\omega^{\log}_{\mathfrak{C}}}} \operatorname{tot} \pi_*\mathcal{V}_2) \cap U$ is contained in $\operatorname{tot}(\bar{A})^\circ$. It follows that $\square \cap Z(\beta)$, which by definition is $Z(\zeta) \cap Z(\beta) \cap \operatorname{tot}(\bar{A})^\circ$, is isomorphic to the open substack of $\operatorname{tot} \pi_*\mathcal{V}$ over $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)^{\circledcirc}_{\omega^{\log}_{\mathfrak{C}}}$ defined by U i.e. it is equal to $\operatorname{tot} \pi_*\mathcal{V}_2$ over $Z(\beta_1) = LG_{g,r}(\mathcal{X}, d)$. This is just $LG_{g,r}(\mathcal{T}, d)$.

To finish the proof, note that by Proposition 3.6.1, intersecting $LG_{g,r}(\mathcal{T}, d)$ with $Z(\alpha)$ gives the degeneracy locus of (3.30) which in this case is $LG_{g,r}(\mathcal{Z}, d)$. This is a closed substack of $LG_{g,r}(\mathcal{X}, d)$ hence proper by Proposition 4.3.2. $\square$

Lemma 5.1.6. *The space $[\square/\mathbb{C}^*_R]$ is a nice quotient stack.*

Proof. Since U is a smooth Deligne–Mumford stack it is in particular a nice quotient stack i.e. $U = [T/H]$ for some noetherian scheme T and reductive linear algebraic group H admitting an ample family of line bundles. Therefore $\bar{A}$ corresponds to an H-equivariant vector bundle E on T. The weight decomposition of $\bar{A}$ coming from the $\mathbb{C}^*_R$-action induces the same decomposition on E. This gives a $\mathbb{C}^*_R$-action on $\operatorname{tot} E$ such that $\operatorname{tot}(\bar{A}) = [\operatorname{tot} E/H \times \mathbb{C}^*_R$. The projection map from $\operatorname{tot} E$ to U is equivariant with respect to the projection map $H \times \mathbb{C}^*_R \to H$. Hence, we can pull back the H-equivariant ample family of line bundles on T to $\operatorname{tot} E$ and twist by all characters of $\mathbb{C}^*_R$ to obtain a $H \times \mathbb{C}^*_R$-equivariant family of ample line bundles on $\operatorname{tot} E$. $\square$

Remark 5.1.7. The two lemmas above insure that the fundamental factorization will provide a well-defined Fourier–Mukai transform (see Diagram 5.6). We use this fact implicitly in the remainder of the article.

5.2. **Independence of choices.** In this section, we show that the various choices made in this construction do not have a large affect on the fundamental factorization. More precisely, two different choices yield two different factorizations related by pushforward (see Definition 5.2.2). In §5.5, we will see that GLSM invariants are thus independent of the choices made.

5.2.1. *Choice of* $\alpha^\vee$. In equation (3.19), the map $\alpha^\vee$ is determined only up to a homotopy. We must therefore check that the PV factorization $\{-\alpha, \beta\}$ depends only on the homotopy class of $\alpha^\vee$. This is proven in [PV16]:

Lemma 5.2.1 (Section 4.3 of [PV16])**.** *Given two homotopic maps* $\alpha^\vee$ *and* $\alpha'^\vee$ *realizing* (3.19)*, there is an induced isomorphism in* $\mathrm{D}([\mathrm{tot}\,\bar{A}/\mathbb{C}^*_R], -\mathrm{ev}^*(\boxplus_{i=1}^r w))$ *between the corresponding PV factorizations* $\{-\alpha, \beta\}$ *and* $\{-\alpha', \beta\}$.

Proof. We summarize the explanation given at the beginning of §4.3 of [PV16]. Namely, a homotopy is a map

$$h : \mathrm{Sym}^{d-2} A \otimes \bigwedge^2 B \to \mathcal{O}_S\,.$$

The endomorphism $\bullet \lrcorner \exp(-h^\vee)$ of the exterior algebra on B gives a Γ-equivariant isomorphism between $\{-\alpha, \beta\}$ and $\{-\alpha', \beta\}$ with inverse $\bullet \lrcorner \exp(h^\vee)$.

□

5.2.2. *Related by pushforward.* The following definition will be used in proving that GLSM invariants do not depend on the various choices of resolutions.

Definition 5.2.2. *We say two factorizations* $K \in \mathrm{D}([\square/\mathbb{C}^*_R], \mathrm{ev}^*(\boxplus_{i=1}^r w))$ *and* $K' \in \mathrm{D}([\square'/\mathbb{C}^*_R], \mathrm{ev}^*(\boxplus_{i=1}^r w))$ *are* related by pushforward *if there exists*

(a) *a set of smooth spaces* $\square_j \to U_j$ *for* $1 \le j \le n$ *and* $\square_{j,j+1} \to U_{j,j+1}$ *for* $1 \le j \le n-1$*, each with* $\mathbb{C}^*_R$*-equivariant evaluation maps* $\mathrm{ev}^i : \square_j \to I[V/\!\!/_\theta G]$ *for each marked point* $1 \le i \le r$*;*

(b) *a set of factorizations*

$$K_j \in \mathrm{D}([\square_j/\mathbb{C}^*_R], -\mathrm{ev}^*(\boxplus_{i=1}^r w)) \text{ and } K_{j,j+1} \in \mathrm{D}([\square_{j,j+1}/\mathbb{C}^*_R], -\mathrm{ev}^*(\boxplus_{i=1}^r w));$$

and

(c) *a commuting diagram of stacks over* $\overline{\mathcal{M}}_{g,r}$ *where all diagonal arrows are closed immersions*

$$\begin{tikzcd}
 & \square_1 \arrow[dl, hook, "f=l_1"'] \arrow[dd] \arrow[dr, hook, "r_1"] & & \cdots \arrow[dl, hook, "l_2"'] \arrow[dr, hook, "r_{n-1}"] & & \square_n \arrow[dl, hook, "l_n"'] \arrow[dd] \arrow[dr, hook, "f'=r_n"] & \\
\square \arrow[dd] & & \square_{1,2} \arrow[dd] & \cdots & \square_{n-1,n} \arrow[dd] & & \square' \arrow[dd] \\
 & U_1 \arrow[dl, hook, "g=\bar{l}_1"'] \arrow[dr, hook, "\bar{r}_1"] & & \cdots \arrow[dl, hook, "\bar{l}_2"'] \arrow[dr, hook, "\bar{r}_{n-1}"] & & U_n \arrow[dl, hook, "\bar{l}_n"'] \arrow[dr, hook, "g'=\bar{r}_n"] & \\
U & & U_{1,2} & \cdots & U_{n-1,n} & & U'
\end{tikzcd}$$

such that there are isomorphisms

$$\mathbb{R}f_*(K_1) \cong K,\ \mathbb{R}f'_*(K_n) \cong K',$$

and

$$\mathbb{R}(r_j)_*(K_j) \cong K_{j,j+1} \cong \mathbb{R}(l_{j+1})_*(K_{j+1}) \text{ for } 1 \le j \le n-1.$$

We will see in §5.5 that if factorizations are related by pushforward, they define the same GLSM invariants. Thus the purpose of this section is to show that the different choices of resolutions made in the two step procedure yield factorizations which are related by pushforward.

In the remainder of this section we will make heavy use of the following proposition, which is proven in [PV16].

Proposition 5.2.3 (Proposition 4.3.1 of [PV16])**.** *Let V be a vector bundle on a smooth global quotient stack $\mathcal{X}$, $w \in \mathrm{H}^0(\mathcal{X}, \mathcal{L})$ a potential, and let $\{-\alpha, \beta\}$ be the Koszul factorization associated to sections $\alpha \in \mathrm{H}^0(\mathcal{X}, V^\vee \otimes \mathcal{L})$ and $\beta \in \mathrm{H}^0(\mathcal{X}, V)$ satisfying $\langle \beta, \alpha \rangle = w$. Let $V_1 \subseteq V$ be a subbundle such that $\beta \bmod V_1$ is a regular section of V/V_1. Assume that the zero locus $\mathcal{X}' = Z(\beta \bmod V_1)$ is smooth and consider the induced sections*

$$\beta' = \beta|_{\mathcal{X}'} \in \mathrm{H}^0(\mathcal{X}', V/V_1) \text{ and } \alpha' = \alpha|_{\mathcal{X}'} \in \mathrm{H}^0(\mathcal{X}', (V/V_1)^\vee).$$

Assume also that either $w|_{\mathcal{X}'}$ is a non-zero-divisor or $w = 0$ and the zero loci $Z(\alpha, \beta)$ and $Z(\alpha', \beta')$ are proper. Then one has an isomorphism

$$\{\alpha, \beta\} \cong \mathbb{R}i_*\{\alpha', \beta'\}$$

in $\mathrm{D}(\mathcal{X}, w)$ where $i : \mathcal{X}' \to \mathcal{X}$ is the inclusion.

5.2.3. *Different choices of evaluation map.* The Koszul factorization K lies in

$$\mathrm{D}([\square/\mathbb{C}^*_R], \mathrm{ev}^*(\boxplus_{i=1}^r w))$$

and thus implicitly depends on the choice of evaluation maps $\mathrm{ev}^i_{\bar{A}}$ of (3.17). In this section, we fix a resolution $[\bar{A} \to \bar{B}]$ and vary the evaluation map, ultimately showing that the result is related by pushforward.

Let $\mathrm{ev}_{\bar{A}} : \bar{A} \to \pi_*(\mathcal{V}|_{\mathscr{G}})$ denote the direct sum of the evaluation maps $\mathrm{ev}^i_{\bar{A}}$ as i ranges from 1 to r. As a map between cochain complexes, $\mathrm{ev}_{\bar{A}}$ is determined only up to homotopy. Given two distinct cochain level realizations $\mathrm{ev}_{\bar{A}}$ and $\mathrm{ev}'_{\bar{A}}$ of the map $[\bar{A} \to \bar{B}] \to \pi_*(\mathcal{V}|_{\mathscr{G}})$ for which the resolution $[\bar{A} \to \bar{B}]$ is admissible, let $\mathrm{ev}_{\bar{A}}$ and $\mathrm{ev}'_{\bar{A}}$ denote the corresponding induced maps $\mathrm{tot}(\bar{A}) \to I[V/G]$. Let $\{-\alpha, \beta\}$ and $\{-\alpha', \beta'\}$ denote the Koszul factorizations in $\mathrm{D}([\mathrm{tot}\,\bar{A}/\mathbb{C}^*], -\mathrm{ev}^*(\boxplus_{i=1}^r w))$ and $\mathrm{D}([\mathrm{tot}\,\bar{A}/\mathbb{C}^*], -\mathrm{ev}'^*(\boxplus_{i=1}^r w))$.

Let Q denote the vector bundle $\pi_*(\mathcal{V}|_{\mathscr{G}})$ over U. For our two choices of cochain level evaluation maps $\mathrm{ev}_{\bar{A}}$ and $\mathrm{ev}'_{\bar{A}}$, there exists a homotopy diagram:

$$\begin{tikzcd}
\bar{A} \arrow[r, "\bar{d}"] \arrow[d, bend right, "\mathrm{ev}'_{\bar{A}}"'] \arrow[d, bend left, "\mathrm{ev}_{\bar{A}}"] & \bar{B} \arrow[dl, "h"] \\
Q &
\end{tikzcd}$$

such that $\mathrm{ev}_{\bar{A}} - \mathrm{ev}'_{\bar{A}} = h \circ \bar{d}$. We extend the above diagram as follows. Let

$$\hat{d} := (\bar{d}, \mathrm{id}) : \widehat{A} := \bar{A} \oplus Q \to \widehat{B} := \bar{B} \oplus Q.$$

Define further the maps $\mathrm{ev}_{\widehat{A}} := (\mathrm{ev}_{\bar{A}}, \mathrm{id})$ and $\mathrm{ev}'_{\widehat{A}} := (\mathrm{ev}'_{\bar{A}}, 0)$ from $\widehat{A}$ to Q. We observe that the map $\widehat{h} := (h, \mathrm{id})$ yields a homotopy: $\mathrm{ev}_{\widehat{A}} - \mathrm{ev}'_{\widehat{A}} = \hat{h} \circ \hat{d}$.

Lemma 5.2.4. *The complex $[\widehat{A} \xrightarrow{\hat{d}} \widehat{B}]$ gives an admissible resolution of $\mathbb{R}\pi_*\mathcal{V}$ with the evaluation map given by either $\mathrm{ev}_{\widehat{A}}$ or $\mathrm{ev}'_{\widehat{A}}$.*

Proof. It is obvious that $[\widehat{A} \xrightarrow{\hat{d}} \widehat{B}] \cong \mathbb{R}\pi_*\mathcal{V}$. We must check that Conditions 1, 2, and 3 are satisfied for the resolution $[\widehat{A} \to \widehat{B}]$ using either evaluation map.

Condition 1 is immediate. To check Conditions 2 and 3, we define the following maps:

$$\widehat{Z} : \mathrm{Sym}\widehat{A} \xrightarrow{\text{proj}} \mathrm{Sym}\bar{A} \xrightarrow{Z} \mathcal{O}_U^r$$

and

$$\widehat{\alpha}^\vee : \mathrm{Sym}\widehat{A} \otimes \widehat{B} \xrightarrow{\text{proj}} \mathrm{Sym}\bar{A} \otimes \bar{B} \xrightarrow{\alpha^\vee} \mathcal{O}_U \,.$$

It is clear that (3.18) commutes with $\widehat{Z}$ and $\widehat{\alpha}^\vee$ replacing Z and $\alpha^\vee$ and that $\widehat{\alpha}^\vee|_{\mathrm{Sym}^{d-1} A\otimes B}$ represents $\mathbb{R}(\pi_d)_* f_d$. Thus Condition 2 holds. Furthermore Condition 3 holds after replacing Z and ev^i_A with $\widehat{Z}$ and $\mathrm{ev}^i_{\widehat{A}}$. This proves that $[\widehat{A} \xrightarrow{\hat{d}} \widehat{B}]$ gives an admissible resolution with respect to $\mathrm{ev}_{\widehat{A}}$.

Recall from the discussion surrounding (3.16), the map $\pi_*\Sigma_{i*}\Sigma_i{}^*\varkappa_w \circ \text{natural} : \mathrm{Sym}Q \to \mathcal{O}_U^r$. Label this map Z_w. Define

$$\widehat{Z}' : \mathrm{Sym}\widehat{A} \xrightarrow{\text{proj}} \mathrm{Sym}\bar{A} \oplus \mathrm{Sym}Q \xrightarrow{Z'+Z_w} \mathcal{O}_U^r \,.$$

It is clear to see that Condition 3 holds after replacing Z and ev^i_A with $\widehat{Z}'$ and $\mathrm{ev}^{i\prime}_{\widehat{A}}$. By quasi-homogeneity, one can easily check using Euler's homogeneous function theorem that there exists a map $\alpha_w^\vee$ such that

$$\begin{array}{ccc} \mathrm{Sym}\, Q & \longrightarrow & \mathrm{Sym}\, Q \otimes Q \\ {\scriptstyle Z_w}\big\downarrow & & {\scriptstyle \alpha_w^\vee}\big\downarrow \\ \mathcal{O}_U^r & \xrightarrow{\text{sum}} & \mathcal{O}_U \end{array} \tag{5.3}$$

commutes. Using this and $\alpha'^\vee$, one can construct a map

$$(\widehat{\alpha}')^\vee : \mathrm{Sym}\widehat{A} \otimes \widehat{B} \to \mathcal{O}_U$$

such that (3.18) commutes after replacing $Z, \alpha^\vee$ with $\widehat{Z}', (\widehat{\alpha}')^\vee$ such that $(\widehat{\alpha}')^\vee|_{\mathrm{Sym}^{d-1} A\otimes B}$ represents $\mathbb{R}(\pi_d)_* f_d$. Thus Condition 2 holds as well. This proves that $[\widehat{A} \xrightarrow{\hat{d}} \widehat{B}]$ gives an admissible resolution with respect to $\mathrm{ev}'_{\widehat{A}}$. $\square$

We will denote by $\widehat{\mathrm{ev}}$ and $\widehat{\mathrm{ev}}'$ the corresponding geometric evaluation maps from $\mathrm{tot}(\widehat{A})$ to $I[V/G]$. Let $\{-\widehat{\alpha}, \widehat{\beta}\}$ respectively $\{-\widehat{\alpha}', \widehat{\beta}'\}$ denote the PV factorizations, in $\mathrm{D}([\mathrm{tot}\,\widehat{A}/\mathbb{C}^*], -\widehat{\mathrm{ev}}^*(\boxplus_{i=1}^r w))$ respectively $\mathrm{D}([\mathrm{tot}\,\widehat{A}/\mathbb{C}^*], -\widehat{\mathrm{ev}}'^*(\boxplus_{i=1}^r w))$, defined by the admissible resolution $[\widehat{A} \to \widehat{B}]$ and the maps $\mathrm{ev}_{\widehat{A}}$ and $\widehat{\alpha}^\vee$ respectively $\mathrm{ev}'_{\widehat{A}}$ and $(\widehat{\alpha}')^\vee$.

The map from $\widehat{A}$ to A_1 is given by the projection to $\bar{A}$ composed with the map $\bar{A} \to A_1$. The locus $\mathrm{tot}\,\bar{A}^\circ$ (defined just before Definition 5.1.1) depends on which evaluation map is used, thus so does the definition of $\square$. Let $\square$ and $\square'$ denote the two loci in $\mathrm{tot}\,\bar{A}$ given by Definition 5.1.1, using the evaluation maps $\mathrm{ev}_{\bar{A}}$ and $\mathrm{ev}'_{\bar{A}}$ respectively. Applying Definition 5.1.1 to $[\widehat{A} \to \widehat{B}]$, we see the the corresponding loci in $\mathrm{tot}\,\widehat{A}$ are equal to $\square \times \mathrm{tot}\, Q$ and $\square' \times \mathrm{tot}\, Q$ respectively. Let i and i' denote the inclusions of $\square$ and $\square'$ into $\square \times \mathrm{tot}\, Q$ and $\square' \times \mathrm{tot}\, Q$.

Lemma 5.2.5. *We have the following relations:*

$$\mathbb{R}i_*\{-\alpha, \beta\} \cong \{-\widehat{\alpha}, \widehat{\beta}\} \text{ and } \mathbb{R}i'_*\{-\alpha', \beta'\} \cong \{-\widehat{\alpha}', \widehat{\beta}'\}.$$

Proof. This is a direct application of Proposition 5.2.3. In this case B is viewed as a sub-bundle of $\widehat{B}$ via the natural inclusion. Note that $\widehat{\beta}$ mod B is simply the tautological section to Q and so trivially a regular section. $\square$

Define

$$L := \ker(\mathrm{ev}_{\widehat{A}} - \mathrm{ev}'_{\widehat{A}}) \text{ and } M := \ker(\hat{h}).$$

Since both $\mathrm{ev}_{\widehat{A}} - \mathrm{ev}'_{\widehat{A}}$ and $\hat{h}$ are clearly surjective over all of U, L and M are both vector bundles. Consider the inclusion $\operatorname{tot} L \hookrightarrow \operatorname{tot} \widehat{A}$. Since the evaluation maps $\widehat{\mathrm{ev}}$ and $\widehat{\mathrm{ev}}'$ are equal on $\operatorname{tot} L$, the inclusion induces pushforwards to both $\mathrm{D}([\operatorname{tot} \widehat{A}/\mathbb{C}^*], -\widehat{\mathrm{ev}}^*(\boxplus_{i=1}^r w))$ and $\mathrm{D}([\operatorname{tot} \widehat{A}/\mathbb{C}^*], -\widehat{\mathrm{ev}}'^*(\boxplus_{i=1}^r w))$.

Let $(\operatorname{tot} L)^\circ$ denote the open locus where $\widehat{\mathrm{ev}}|_L$ maps to $(I\mathcal{T})^r$, where we recall $\mathcal{T} = V/\!\!/_\theta G \hookrightarrow [V/G]$. Let $\square_L$ denote the locus $\{\zeta = 0\} \subseteq (\operatorname{tot} L)^\circ$ of Definition 5.1.1. It is easy to check that the map $L \to \widehat{A} \to A_1$ is still surjective, so $\square_L$ is smooth.

Let $j : \square_L \to \square \times \operatorname{tot} Q$ and $j' : \square_L \to \square' \times \operatorname{tot} Q$ denote the inclusions.

Lemma 5.2.6. *The complex $[L \to M]$ gives an admissible resolution of $\mathbb{R}\pi_*\mathcal{V}$. The corresponding PV factorization $\{-\alpha'', \beta''\}$ satisfies the following:*

$$\mathbb{R}j_*\{-\alpha'', \beta''\} \cong \{-\widehat{\alpha}, \widehat{\beta}\}$$
$$\mathbb{R}j'_*\{-\alpha'', \beta''\} \cong \{-\widehat{\alpha}', \widehat{\beta}'\}.$$

Proof. It is an easy exercise to check that $[L \to M]$ is quasi-isomorphic to $[\widehat{A} \to \widehat{B}]$ and therefore to $\mathbb{R}\pi_*\mathcal{V}$. We define the evaluation map $L \to Q$ to simply be the restriction of $\widehat{\mathrm{ev}}$ or $\widehat{\mathrm{ev}}'$ (they are equal on L by construction). To check surjectivity consider the following. Given $p \in Q$, choose $a \in \bar{A}$ such that $\bar{\mathrm{ev}}'(a) = p$. This is possible since $\bar{\mathrm{ev}}'$ is surjective. Let $p' = \bar{\mathrm{ev}}(a) - p$. Then $\widehat{\mathrm{ev}}(a,p) - \widehat{\mathrm{ev}}'(a,p) = \bar{\mathrm{ev}}(a) + p' - \bar{\mathrm{ev}}'(a) = 0$. Therefore $(a,p) \in \widehat{A}$ actually lies in L, and $\widehat{\mathrm{ev}}'(a,p') = p$. This shows that Condition 1 is satisfied. To verify Condition 2, we simply restrict the map $\widehat{\alpha}^\vee : \mathrm{Sym}\widehat{A} \otimes \widehat{B} \to \mathcal{O}_U$. Condition 3 is apparent by construction.

Let $\{-\alpha'', \beta''\}$ denote the corresponding PV factorization on $[\square_L/\mathbb{C}^*_R]$. Applying Proposition 5.2.3 to each of $\{-\widehat{\alpha}, \widehat{\beta}\}$ and $\{-\widehat{\alpha}', \widehat{\beta}'\}$ yields the result. In the notation of Proposition 5.2.3, V is given by $\widehat{B}$ and V_1 is M. The map $\bar{\beta} \mod M$ is equal to the section $\mathrm{ev}_{\widehat{A}} - \mathrm{ev}'_{\widehat{A}}$ defining $\operatorname{tot} L$ inside $\operatorname{tot} \widehat{A}$ (and $\square_L$ inside both $\square \times Q$ and $\square' \times Q$), and therefore regular. $\square$

We arrive at the desired statement.

Proposition 5.2.7. *The factorizations $\{-\alpha, \beta\}$ and $\{-\alpha', \beta'\}$ are related by pushforward.*

Proof. By the previous two lemmas we have $\mathbb{R}i_*\{-\alpha, \beta\} \cong \{-\widehat{\alpha}, \widehat{\beta}\} \cong \mathbb{R}j_*\{-\alpha'', \beta''\}$ and $\mathbb{R}i_*\{-\alpha', -\beta'\} \cong \{-\widehat{\alpha}', \widehat{\beta}'\} \cong \mathbb{R}j'_*\{-\alpha'', \beta''\}$. The conclusion follows. $\square$

5.2.4. *Different choices of resolution.* We next consider the choice of resolution $[\bar{A} \to \bar{B}]$ over U.

Lemma 5.2.8. *Given two different choices of admissible resolutions $[\bar{A} \to \bar{B}]$ and $[\bar{A}' \to \bar{B}']$ of $\mathbb{R}\pi_*(\mathcal{V})$ over U, the factorizations $\{\alpha, \beta\}$ and $\{\alpha', \beta'\}$ are related by pushforward.*

Proof. By Lemma 3.6.5, there exists a roof diagram realizing the quasi-isomorphism between $[\bar{A} \to \bar{B}]$ and $[\bar{A}' \to \bar{B}']$ such that the roof is a two term complex $[\bar{A}'' \to \bar{B}'']$ of vector bundles and all maps are surjective. By Lemma 3.4.3, the roof provides another admissible resolution. The evaluation map on $\bar{A}''$ may be induced either by the map to $\bar{A}$ or the map to $\bar{A}'$. These may not agree, however by the previous section the corresponding factorizations are related by pushforward.

Thus we reduce to the situation that there exists a morphism of cochain complexes $[\bar{A} \to \bar{B}] \to [\bar{A}' \to \bar{B}']$ realizing the quasi-isomorphism, and such that $[\bar{A} \to \bar{B}]$ is admissible and the evaluation map $\mathrm{ev}_{\bar{A}}$ from $\bar{A}$ factors through $\bar{A}'$. Up to homotopy, this map decomposes as

$$[\bar{A} \to \bar{B}] \to [\bar{A}' \oplus \bar{B} \to \bar{B}' \oplus \bar{B}] \to [\bar{A}' \to \bar{B}'].$$

The map $\bar{A} \to \bar{A}' \oplus \bar{B}$ is injective due to the fact that $[\bar{A} \to \bar{B}] \to [\bar{A}' \to \bar{B}']$ is a quasi-isomorphism. The second map has (up to homotopy) a right inverse consisting of injections, thus we may assume without loss of generality that the maps of vector bundles in

$$[\bar{A} \to \bar{B}] \to [\bar{A}' \to \bar{B}']$$

are injective. Because the map $\bar{A} \to A_1$ factors through $\bar{A}'$, as does the evaluation map $\mathrm{ev}_{\bar{A}}$, the closed immersion $\mathrm{tot}(A) \to \mathrm{tot}(\bar{A}')$ induces an immersion $\square \to \square'$. The section $\beta' \mod \bar{B}$ defines the locus $\square \subseteq \square'$, therefore by Proposition 5.2.3, the Koszul factorization $\{-\alpha', \beta'\}|_{\square}$ is isomorphic to the pushforward of $\{-\alpha, \beta\}|_{\square'}$ under the map $\square \to \square'$. $\square$

The resolution $[A_1 \xrightarrow{d_1} B_1]$ of $\mathbb{R}\pi_*(\mathcal{V}_1)$ in Lemma 4.2.4 and consequently the closed immersion of $LG_{g,r}(\mathcal{X}, d) \to \overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d)$ in Proposition 4.3.2 depend on a choice of closed immersion from the universal curve $\underline{\mathcal{C}} \subseteq \mathbb{P}^{N-1}$ (recall § 4.2). We next show that two fundamental factorizations coming from different choices of this immersion are related by pushforward.

Consider two different resolutions $i : \underline{\mathcal{C}} \to \mathbb{P}^{N-1}$ and $i' : \underline{\mathcal{C}} \to \mathbb{P}^{M-1}$. Pulling back the first two terms of the Euler sequence on $\mathbb{P}^{N-1}$ resp. $\mathbb{P}^{M-1}$ to $\mathfrak{C}_{BG_1}$ yields two vector bundles with sections:

$$\mathcal{O}_{\mathfrak{C}_{BG_1}} \to (\mathcal{N}_{BG_1})^{\oplus N} \text{ resp. } \mathcal{O}_{\mathfrak{C}_{BG_1}} \to (\mathcal{M}_{BG_1})^{\oplus M}.$$

Tensoring the two yields a section of $(\mathcal{N}_{BG_1})^{\oplus N} \otimes (\mathcal{M}_{BG_1})^{\oplus M}$ which defines the map $\mathfrak{C}_{BG_1} \to \mathbb{P}^{N+M-1}$ coming from the Segre embedding $\mathbb{P}^{N-1} \times \mathbb{P}^{M-1} \to \mathbb{P}^{N+M-1}$. Note the section factors as

$$\mathcal{O}_{\mathfrak{C}_{BG_1}} \to \mathcal{N}_{BG_1}^{\oplus N} \to (\mathcal{N}_{BG_1})^{\oplus N} \otimes {\mathcal{M}_{BG_1}}^{\oplus M}.$$

Let $N' := N + M$ and let $\mathcal{N}'_{BG_1} := \mathcal{N}_{BG_1} \otimes \mathcal{M}_{BG_1}$. Tensoring the above inclusions by $\mathcal{V}_1$ yields the following map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{V}_1 \arrow[r] \arrow[d, "\mathrm{id}"] & \mathcal{V}_1 \otimes \mathcal{N}_{BG_1}^{\oplus N} \arrow[r] \arrow[d, hook] & \mathcal{Q} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{V}_1 \arrow[r] & \mathcal{V}_1 \otimes (\mathcal{N}'_{BG_1})^{\oplus N'} \arrow[r] & \mathcal{Q}' \arrow[r] & 0
\end{tikzcd} \tag{5.4}$$

where $\mathcal{Q}$ and $\mathcal{Q}'$ are defined as the respective cokernels. In the notation of Lemma 4.2.4, let $A_1 := \mathbb{R}\pi_*\mathcal{V}_1 \otimes \mathcal{N}_{BG_1}^{\oplus N}$ resp. $A'_1 := \mathcal{V}_1 \otimes (\mathcal{N}'_{BG_1})^{\oplus N'}$ and $B_1 := \mathbb{R}\pi_*\mathcal{Q}$ resp. $B'_1 := \mathbb{R}\pi_*\mathcal{Q}'$.

Following the embedding construction of § 4.2 and § 4.3 with each of the two resolutions of $\mathcal{V}_1$ from (5.4), we obtain open substacks $U \subset \mathrm{tot}(A_1)$ resp. $U' \subset \mathrm{tot}(A'_1)$ over $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)^{\circ}_{\omega^{\log}_{\mathfrak{C}}}$ such that all the properties of Theorem 4.3.4 hold. Furthermore, from the inclusion $A_1 \hookrightarrow A'_1$ obtained from (5.4), one checks that there is an induced morphism $U \hookrightarrow U'$. By replacing U' with a smaller open subset if necessary, we may assume we are in the following situation.

Reduction 5.2.9. . Given two resolutions $[A_1 \to B_1]$ and $[{A_1}' \to {B_1}']$ of $\mathbb{R}\pi_*(\mathcal{V}_1)$ over $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)^{\circ}_{\omega^{\log}_{\mathfrak{C}}}$, constructed as in § 4.2, we can assume without loss of generality that there exists a quasi-isomorphism

$$f\colon [A_1 \to B_1] \to [{A_1}' \to {B_1}']$$

realized at the cochain level, such that the maps $f_{A_1}\colon A_1 \to {A_1}'$ and $f_{B_1}: B_1 \to {B_1}'$ are injective, and $U \subseteq \mathrm{tot}(A_1)$ is the preimage of $U' \subseteq \mathrm{tot}\, {A_1}'$ under f. We can assume further that there exist admissible resolutions on both U and U'.

Lemma 5.2.10. *With the situation given as above, there exist admissible resolutions $[\bar{A} \to \bar{B}]$ of $\mathbb{R}\pi_*(\mathcal{V})$ over U and $[\bar{A}' \to \bar{B}']$ of $\mathbb{R}\pi_*(\mathcal{V})$ over U' together with a closed immersion $\bar{f} : \mathrm{tot}(\bar{A}) \to \mathrm{tot}(\bar{A}')$ such that the diagram*

$$\begin{tikzcd} \square \arrow[r, hook, "\bar{f}"] \arrow[d] & \square' \arrow[d] \\ U \arrow[r, hook, "f"] & U' \end{tikzcd}$$

commutes, and the pushforward $\mathbb{R}\overline{f}_*(\{-\alpha, \beta\})$ is isomorphic to $\{-\alpha', \beta'\}$ in the derived category of factorizations.

Proof. By abuse of notation, we let $A_1, {A_1}', B_1, {B_1}'$ denote the pulled-back vector bundles over U'. Let $\bar{A}' \to \bar{B}'$ denote a resolution of $\mathbb{R}\pi_*(\mathcal{V})$ over U' which surjects onto $[{A_1}' \to {B_1}']$. Let C denote the cokernel of $f : A_1 \to {A_1}'$. We have the following commutative diagram:

$$\begin{tikzcd}
& \bar{B} \arrow[rr, hook, "\bar{g}"] \arrow[dd, two heads] & & \bar{B}' \arrow[rr] \arrow[dd, two heads] & & C \arrow[dd, "\cong"] \\
\bar{A} \arrow[ur] \arrow[rr, hook, "\bar{f}"] \arrow[dd, two heads, "\rho"] & & \bar{A}' \arrow[ur, "\bar{d}'"] \arrow[rr, "\bar{\gamma}"] \arrow[dd, two heads, "\rho'"] & & C \arrow[ur, "\cong"] \arrow[dd, two heads, "\cong"] & \\
& B_1 \arrow[rr, hook, "g"] & & {B_1}' \arrow[rr] & & C \\
A_1 \arrow[ur] \arrow[rr, hook, "f"] & & A_1' \arrow[ur, "d'"] \arrow[rr, "\gamma"] & & C \arrow[ur, "\cong"] &
\end{tikzcd}$$

where $\bar{\gamma}$ is the composition $\bar{A}' \to {A_1}' \to C$, $\bar{A} = \ker(\bar{\gamma})$, and similarly $\bar{B}$ is the kernel of the composition from $\bar{B}'$ to C. Note that $\bar{A}$ surjects onto A_1, and that $[\bar{A} \to \bar{B}]$ is quasi-isomorphic to $[\bar{A}' \to \bar{B}']$.

Let $\mathrm{tot}(\bar{A}|_U)$ and $\mathrm{tot}(\bar{A}')$ denote the total spaces of $\bar{A}$ and $\bar{A}'$ over U and U' respectively. Let $p : T \to U$ and $p' : T' \to U'$ denote the projections. Recall that $p^*(A_1)$ and ${p'}^*(A_1')$ have sections

$$\zeta = p^*(\rho \circ \mathrm{taut}_{\bar{A}}) - p^*(\mathrm{taut}_{A_1})$$

and

$$\zeta' = {p'}^*(\rho' \circ \mathrm{taut}_{\bar{A}'}) - {p'}^*(\mathrm{taut}_{A_1'})$$

whose zero loci define $\square$ and $\square'$ respectively. The above diagram shows the map $\mathrm{tot}(\bar{A}|_U) \to \mathrm{tot}(\bar{A}')$ induced by $\bar{f}$ sends $\square$ into $\square'$. In fact $\square$ is the zero locus of the section ${p'}^*(\gamma \circ \mathrm{taut}_{A_1'}) \in \Gamma(\square', {p'}^*(C))$.

On $\square'$ we have the obstruction bundle $E' = p'^*(\bar{B}')$ which contains $E = p'^*(\bar{B})$ as a sub-bundle. E' has a section $\beta' = \bar{d}' \circ \mathrm{taut}_{\bar{A}'}$. Note that $E'/E = p'^*(\bar{B}'/\bar{B}) = p'^*(C)$ and $\beta'\mathrm{mod}(E)$ is given by

$$\beta'\mathrm{mod}(E) = p'^*(\bar{\gamma} \circ \mathrm{taut}_{\bar{A}'}) = p'^*(\gamma \circ \mathrm{taut}_{A_1'}),$$

where the last equality holds since we are on $\square'$. By Proposition 5.2.3, the claim follows. $\square$

Proposition 5.2.11. *Any two Koszul factorizations*

$$\{-\alpha, \beta\} \in \mathrm{D}([\square/\mathbb{C}^*_R], -\mathrm{ev}^*(\boxplus_{i=1}^r w)) \text{ and } \{-\alpha', \beta'\} \in \mathrm{D}([\square'/\mathbb{C}^*_R], -\mathrm{ev}^*(\boxplus_{i=1}^r w))$$

constructed via the two-step procedure are related by pushforward.

Proof. Given two resolutions $[A_1 \xrightarrow{d_1} B_1]$ and $[A_1' \xrightarrow{d_1'} B_1{}']$ of $\mathbb{R}\pi_*(\mathcal{V}_1)$ over $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)^\circ_{\omega^{\log}_{\mathfrak{C}}}$ and corresponding open sets $U \subseteq \mathrm{tot}(A_1)$ and $U' \subseteq \mathrm{tot}(A_1{}')$, by Reduction 5.2.9 we can assume that these two complexes are related by a quasi-isomorphism

$$f : [A_1 \to B_1] \to [A_1{}' \to B_1{}']$$

such that

- the maps $f_A : A_1 \to A_1{}'$ and $f_B : B_1 \to B_1{}'$ are injective; and
- $U \subseteq \mathrm{tot}(A_1)$ is the preimage of $U' \subseteq \mathrm{tot}\, A_1{}'$ under f.

In this particular case, by Lemma 5.2.10 there exist resolutions $[\bar{A} \to \bar{B}]$ of $\mathbb{R}\pi_*(\mathcal{V})$ over U and $[\bar{A}' \to \bar{B}']$ of $\mathbb{R}\pi_*(\mathcal{V})$ over U' together with a closed immersion $\bar{f} : \mathrm{tot}(\bar{A}) \to \mathrm{tot}(\bar{A}')$ such that the factorization $\{-\alpha, \beta\}$ on $\square$ is equivalent to the factorization $\{-\alpha', \beta'\}$ on $\square'$ by pushforward under $\bar{f}|_\square$.

Thus we reduce to the case that U is fixed. Then by Lemma 5.2.8 we conclude that the factorizations are related by pushforward. $\square$

5.3. **Rigidified evaluation and the state space.**

5.3.1. *Rigidified evaluation.* Recalling Definition 5.1.1 there exist $\mathbb{C}^*_R$-equivariant evaluation maps

$$\mathrm{ev}^i \colon \square \to I\mathcal{T} = \coprod_{(g)} [(V^{ss}(\theta))^g)/C_G(g)] \tag{5.5}$$

where (g) runs over all conjugacy classes in G. For each conjugacy class (g), we choose a representative g; then $C_G(g)$ denotes the centralizer of g in G, and $(V^{ss}(\theta))^g$ denotes the points of $V^{ss}(\theta)$ which are fixed by g. These maps combine to form a map

$$\mathrm{ev} \colon \square \to (I\mathcal{T})^r = \coprod_{(\underline{g})} \left[(\bigoplus_{i=1}^r (V^{ss}(\theta))^{g_i} / \prod_{i=1}^r C_G(g_i) \right].$$

where $(\underline{g})$ runs over all ordered sets of r conjugacy classes $(g_1), \ldots, (g_r)$.

As described in §3.7, the map ev is $\mathbb{C}^*_R$-equivariant with respect to the natural action on $\mathrm{tot}\, A$ and the diagonal action of $\mathbb{C}^*_R$ on V^r. Thus it induces a map (which we will also denote by ev) from $[\mathrm{tot}\, A/\mathbb{C}^*_R]$ to the quotient of $(I[V /\!\!/_\theta G])^r$ by

$\mathbb{C}^*_R$. Also, notice that the action of $G^r \times \mathbb{C}^*_R$ has a generic stabilizer given by $\langle J\rangle$. $\langle J\rangle = G \cap \mathbb{C}^*_R \subseteq \mathrm{Gl}(V)$. This gives an exact sequence,

$$\begin{array}{ccccccccc} 1 & \to & \langle J\rangle & \to & G^r \times \mathbb{C}^*_R & \to & \overbrace{\Gamma \times_{\mathbb{C}^*} \cdots \times_{\mathbb{C}^*} \Gamma}^{r-\text{times}} & \to & 1 \\ & & & & (g_1, ..., g_r, t) & \mapsto & (g_1 t, g_2 t, ..., g_r t) & & \end{array}$$

where $\Gamma \times_{\mathbb{C}^*} \cdots \times_{\mathbb{C}^*} \Gamma$ denotes the fiber product over the character χ, that is, $\Gamma \times_{\mathbb{C}^*} \cdots \times_{\mathbb{C}^*} \Gamma$ is the kernel of map $\Gamma \times \cdots \times \Gamma \to \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ sending

$$(\gamma_1, ..., \gamma_r) \mapsto (\chi(\gamma_1)\chi(\gamma_2)^{-1}, \chi(\gamma_2)\chi(\gamma_3)^{-1}, ..., \chi(\gamma_{r-1})\chi(\gamma_r)^{-1}).$$

Analogously,

$$1 \to \langle J\rangle \to \prod_{i=1}^r C_G(g_i) \times \mathbb{C}^*_R \to C_\Gamma(g_1) \times_{\mathbb{C}^*} \cdots \times_{\mathbb{C}^*} C_\Gamma(g_r) \to 1$$

where $C_\Gamma(g) = C_G(g) \cdot \mathbb{C}^*_R$ is the centralizer of g in Γ. Rigidification gives a map,

$$\mathrm{rigidify}\colon [V^r/(G^r \times \mathbb{C}^*_R)] \to [V^r/(\Gamma \times_{\mathbb{C}^*} \cdots \times_{\mathbb{C}^*} \Gamma)].$$

Let $(\underline{\boldsymbol{g}})$ denote an ordered set of r conjugacy classes $(g_1), \ldots, (g_r)$. The map rigidify induces the map

$$I_{\mathrm{rig}}\colon \left[\left(\bigoplus_{i=1}^r (V^{ss}(\theta))^{g_i}\right) / \left(\prod_{i=1}^r C_G(g_i) \times \mathbb{C}^*_R\right)\right] \to \left[\left(\bigoplus_{i=1}^r (V^{ss}(\theta))^{g_i}\right) / \left(C_\Gamma(\underline{\boldsymbol{g}})\right)\right],$$

where $C_\Gamma(\underline{\boldsymbol{g}}) := C_\Gamma(g_1) \times_{\mathbb{C}^*} \cdots \times_{\mathbb{C}^*} C_\Gamma(g_r)$. Note that the source of I_{rig} is simply a component of the quotient of $(I[V/\!\!/_\theta G])^r$ by $\mathbb{C}^*_R$. Combined, these observations allow us to make the following definition.

Definition 5.3.1. *Define the* evaluation map *as the composition of I_{rig} with* ev*.*

$$\mathrm{ev} = I_{\mathrm{rig}} \circ \mathrm{ev}\colon [\square/\mathbb{C}^*_R] \to \coprod_{(\underline{\boldsymbol{g}})} \left[(\bigoplus_{i=1}^r (V^{ss}(\theta))^{g_i} / C_\Gamma(\underline{\boldsymbol{g}}) \right],$$

where the union is over all r-tuples $(\underline{\boldsymbol{g}})$ of conjugacy classes in G.

5.3.2. *The state space of a GLSM.*

Definition 5.3.2. *Define*

$$\mathcal{H}^{\mathrm{ext}}_{(g)} := \mathrm{HH}_*\left([(V^{ss}(\theta))^g / C_\Gamma(g)], w\right).$$

The extended GLSM state space *is defined to be $\mathcal{H}^{\mathrm{ext}} := \bigoplus_{(g)} \mathcal{H}^{\mathrm{ext}}_{(g)}$ where (g) runs over all conjugacy classes in G.*

Remark 5.3.3. To better understand the target of the evaluation map ev, note that

$$\begin{aligned} \mathrm{HH}_*\left(\left[\left(\bigoplus_{i=1}^r (V^{ss}(\theta))^{g_i} / C_\Gamma(\underline{\boldsymbol{g}})\right], \boxplus_{i=1}^r w\right) &= \bigotimes_{i=1}^r \mathrm{HH}_*\left([(V^{ss}(\theta))^{g_i} / C_\Gamma(g_i)], w\right) \\ &= \bigotimes_{i=1}^r \mathcal{H}^{\mathrm{ext}}_{(g_i)} \\ &\subseteq (\mathcal{H}^{\mathrm{ext}})^{\otimes r}. \end{aligned}$$

5.4. **The restricted state space.** In fact we will deal most often with a subspace of $\mathcal{H}^{\text{ext}}$ called the restricted state space and denoted by

$$\mathcal{H}^{res} \subseteq \mathcal{H}^{\text{ext}},$$

similarly to [PV16, Equation (5.13)]. The definition and properties of the restricted state space are not strictly necessary for the results of this paper, thus we will save the bulk of the discussion for the sequel [CFG+]. We record here only a brief summary of the key facts for completeness of exposition.

Proposition 5.4.1 ([CFG+])**.** *The vector space $\mathcal{H}^{\text{ext}}_{(g)}$ decomposes as a direct sum*

$$\mathcal{H}^{\text{ext}}_{(g)} = \bigoplus_{(h)} e_{(h)}(\mathcal{H}^{\text{ext}}_{(g)}),$$

where (h) runs over all conjugacy classes of G.

Definition 5.4.2. *Define the restricted state space to be*

$$\mathcal{H}^{res} := \bigoplus_{(g)} e_{(id)}(\mathcal{H}^{\text{ext}}_{(g)}),$$

where (id) denotes the identity conjugacy class. Thus on each twisted sector $[(V^{ss}(\theta))^g/C_\Gamma(g)]$, $\mathcal{H}^{res}$ contains the piece of $\mathcal{H}^{\text{ext}}_{(g)}$ corresponding to the identity under the decomposition of proposition 5.4.1.

Remark 5.4.3. In the following paper [CFG+], we will define a pairing on the restricted state space, as is necessary for the definition of a cohomological field theory.

Remark 5.4.4. In the special case of a GLSM $(V, G, \mathbb{C}^*_R, \theta, w)$ where $[V/\!\!/_\theta G]$ is affine, Proposition 5.4.1 was proven by Polishchuk–Vaintrob ([PV16], Theorem 2.6.1).

Remark 5.4.5. Note that when $[V/\!\!/_\theta G]$ is a smooth variety, $\mathcal{H}^{\text{ext}}$ and $\mathcal{H}^{res}$ are equal.

5.5. **GLSM invariants.** By Definition 5.3.1 there exists an evaluation map

$$\text{ev}\colon [\square/\mathbb{C}^*_R] \to \coprod_{(\underline{\boldsymbol{g}})} \left[(\bigoplus_{i=1}^{r} (V^{ss}(\theta))^{g_i}/C_\Gamma(\underline{\boldsymbol{g}}) \right],$$

where the union is over all r-tuples $(\underline{\boldsymbol{g}})$ of conjugacy classes in G.

Fix a resolution $\widetilde{U}$ of singularities of the Deligne–Mumford stack $\overline{U}$ (which exists due to a general result by Temkin [Tem09, Theorem 5.1.1]), where $\overline{U}$ is the closure of $U \subseteq LG^{eq}_{g,r}(\mathcal{Y}, d')$ (see Theorem 4.3.4). The space $\widetilde{U}$ is a smooth and proper Deligne–Mumford stack. Denote by $\widetilde{p}\colon \square \to \widetilde{U}$ the composition of the projection $p\colon \square \to U$ followed by the open immersion $U \to \widetilde{U}$.

We use $K = K_{g,r,d}$ to define an integral transform $\Phi_{K_{g,r,d}}$ by the following diagram.

$$\begin{tikzcd}
\mathrm{D}([\square/\mathbb{C}^*_R], \mathrm{ev}^*(\boxplus_{i=1}^r w)) \arrow[r, "-\overset{\mathbb{L}}{\otimes} K_{g,r,d}"] & \mathrm{D}([\square/\mathbb{C}^*_R], 0)_{LG_{g,r}(\mathcal{Z},d)} \arrow[d, "\mathbb{R}\widetilde{p}_*"] \\
\mathrm{D}(\coprod_{(\underline{\boldsymbol{g}})} \left[\left(\oplus_{i=1}^r (V^{ss}(\theta))^{g_i}/C_\Gamma(\underline{\boldsymbol{g}})\right], \boxplus_{i=1}^r w\right) \arrow[u, "\mathbb{L}\,\mathrm{ev}^*"] & \mathrm{D}([\widetilde{U}/\mathbb{C}^*_R], 0) \simeq \mathrm{D}([\widetilde{U}/\mu_{\mathbf{deg}}])
\end{tikzcd} \tag{5.6}$$

Note that by Lemma 5.1.5, tensoring with $K_{g,r,d}$ gives a factorization supported on $LG_{g,r}(\mathcal{Z}, d)$. Since the action of $\mathbb{C}^*_R$ on $LG_{g,r}(\mathcal{Z}, d)$ and on $\widetilde{U}$ is trivial, $[LG_{g,r}(\mathcal{Z}, d)/\mathbb{C}^*_R]$ is proper over $[\widetilde{U}/\mathbb{C}^*_R]$. Thus the pushforward $\mathbb{R}\widetilde{p}_*$ is well-defined.

The equivalence
$$\mathrm{D}([\widetilde{U}/\mathbb{C}^*_R], 0) \cong \mathrm{D}([\widetilde{U}/\mu_{\mathbf{deg}}])$$
is [PV16, Proposition 1.2.2]. Pushing forward to Hochschild homology yields a map
$$(\Phi_{K_{g,r,d}})_* : (\mathcal{H}^{\mathrm{ext}})^{\otimes r} \to \mathrm{HH}_*([\widetilde{U}/\mu_{\mathbf{deg}}]).$$
We further pull back by the quotient map
$$q : \widetilde{U} \to [\widetilde{U}/\mu_{\mathbf{deg}}]$$
to obtain a map to $\mathrm{HH}_*(\widetilde{U})$.

Given an r-tuple of conjugacy classes $(\underline{\boldsymbol{g}}) = (g_1), \ldots, (g_r)$, let $\square_{(\underline{\boldsymbol{g}})}$ denote the open and closed substack of $\square$ such that $\mathrm{ev}\,|_{\square_{(\underline{\boldsymbol{g}})}}$ maps to $\left[\left(\bigoplus_{i=1}^r (V^{ss}(\theta))^{g_i}/C_\Gamma(\underline{\boldsymbol{g}})\right]\right.$, and let $\widetilde{U}_{(\underline{\boldsymbol{g}})}$ be the open and closed substack of $\widetilde{U}$ which $\square_{(\underline{\boldsymbol{g}})}$ maps into via $\widetilde{p}$. Denote by m_i the order of g_i. Then $1/(m_1 \cdots m_r)$ is the degree of the map
$$\overline{\mathcal{M}}_{g,r}\left(\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma}/\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d), F\right) \to \overline{\mathcal{K}}_{g,r}\left(\mathfrak{C}_{\mathcal{X}^{\mathrm{rig}},\Gamma}/\overline{\mathcal{K}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, d), F\right)$$
from Remark 4.3.1 after restricting to the open and closed subset indexed by $(\underline{\boldsymbol{g}})$. Define the map $\mathrm{ord}_{(\underline{\boldsymbol{g}})} : \mathrm{HH}_*(\widetilde{U}_{(\underline{\boldsymbol{g}})}) \to \mathrm{HH}_*(\widetilde{U}_{(\underline{\boldsymbol{g}})})$ to be multiplication by $(m_1 \cdots m_r)$, and let
$$\mathrm{ord} : \mathrm{HH}_*(\widetilde{U}) \to \mathrm{HH}_*(\widetilde{U})$$
be the direct sum of $\mathrm{ord}_{(\underline{\boldsymbol{g}})}$ over all choices of $(\underline{\boldsymbol{g}})$. Because our moduli spaces have been constructed to have sections at the marked point gerbes, this correction factor is necessary to obtain the correct invariants. This phenomena arises in orbifold Gromov–Witten theory as well (see [AGV08, §6.1.3]).

Recall we denote by ϕ_{HKR} the isomorphism from $\mathrm{HH}_*(X)$ to $\mathrm{H}^*(X)$ for X any smooth projective variety and for $\mathcal{X}$ a smooth projective Deligne–Mumford quotient stack we denote by
$$\bar{\phi}_{\mathrm{HKR}} \colon \mathrm{HH}_*(\mathcal{X}) \to \mathrm{H}^*(\mathcal{X})$$
the HKR *morphism* (not an isomorphism unless $\mathcal{X}$ is a variety) of Definition 2.4.5.

Definition 5.5.1. *Let $(V, G, \mathbb{C}^*_R, \theta, w)$ be a convex hybrid model. For g, r satisfying $2g - 2 + r > 0$ and $d \in Hom_{\mathbb{Z}}(\widehat{G}, \mathbb{Q})$, define $\Lambda_{g,r,d} : (\mathcal{H}^{res})^{\otimes r} \to \mathrm{H}^*(\overline{\mathcal{M}}_{g,r})$ as the following map*

$$\Lambda_{g,r,d}(s_1, \ldots, s_r) := \mathrm{proj}_*\left(\frac{\mathrm{td}(T_{\widetilde{U}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}})}{\mathrm{td}(\mathbb{R}\pi_*\mathcal{V})} \cup \left(\bar{\phi}_{\mathrm{HKR}} \circ \mathrm{ord} \circ q^* \circ (\Phi_{K_{g,r,d}})_*(s_1, \ldots, s_r)\right)\right)$$

where proj *is the projection to* $\overline{\mathcal{M}}_{g,r}$ *and* $s_1, \ldots, s_r$ *are elements of* $\mathcal{H}^{res}$.

The set $\{\Lambda_{g,r,d}\}$ *for all choices of* g, r *and* d *will be referred to as the GLSM invariants of* $(V, G, \mathbb{C}^*_R, \theta, w)$.

Remark 5.5.2. The pullback of $T_{\widetilde{U}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}}$ to U is the vector bundle A_1.

Remark 5.5.3. In the second paper [CFG+], we will show that the collection $\{\Lambda_{g,r,d}\}$ defines a cohomological field theory.

Theorem 5.5.4. *Given a hybrid model GLSM* $(V, G, \mathbb{C}^*_R, \theta, w)$, *convex over* BG_1, *the invariants* $\Lambda_{g,r,d}$ *are independent of the choice of resolutions.*

Proof. Let $\square' \to U'$ and $\square \to U$ denote two instances of the two-step procedure of the previous section for different choices of resolutions. Let K and K' denote the factorizations in $\square$ and $\square'$ respectively, and let $\Lambda_{g,r,d}$ and $\Lambda'_{g,r,d}$ denote the corresponding sets of invariants for $(V, G, \mathbb{C}^*_R, \theta, w)$. By Proposition 5.2.11, we can assume that

$$\mathbb{R}f_*(K) = K' \tag{5.7}$$

More precisely, by Reduction 5.2.9, we can assume without loss of generality that there exists a closed immersion

$$f : U \to U'.$$

Let $\widetilde{U}$ and $\widetilde{U}'$ denote resolutions of the closures $\overline{U}$ and $\overline{U}'$. These can be constructed so that there exists a map $\tilde{f} : \widetilde{U} \to \widetilde{U}'$. To see this, first choose a resolution $\widetilde{U}'$ of $\overline{U}'$. The map f can be assumed to extend to a map $\bar{f} : \overline{U} \to \overline{U}'$. If

$$\overline{U} \times_{\overline{U}'} \widetilde{U}'$$

is smooth we let this define $\widetilde{U}$, otherwise we resolve this space to define $\widetilde{U}$. By (5.7) we see that

$$\tilde{f}_* \circ (\Phi_K)_* = (\Phi_{K'})_*. \tag{5.8}$$

We then observe that

$$\begin{aligned} &\tilde{f}_*\left(\mathrm{td}\left(T_{\widetilde{U}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}}\right)\bar{\phi}_{\mathrm{HKR}}(-)\right) \\ =&\mathrm{td}\left(T_{\widetilde{U}'/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}}\right)\tilde{f}_*\left(\mathrm{td}(T_{\widetilde{U}/\widetilde{U}'})\bar{\phi}_{\mathrm{HKR}}(-)\right) \\ =&\mathrm{td}\left(T_{\widetilde{U}'/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}}\right)\bar{\phi}_{\mathrm{HKR}}\tilde{f}_*(-) \end{aligned} \tag{5.9}$$

where the last equality follows from the same argument as Theorem 2.5.1. Combining (5.8) and (5.9) implies the result. $\square$

6. Comparisons with other constructions

6.1. **Comparison with Gromov–Witten theory and cosection localization.** In this section we compare the above GLSM invariants for a *geometric phase* with various Gromov–Witten type invariants defined via a virtual fundamental class.

6.1.1. *The virtual cycle for GLSM invariants in a geometric phase.* For all of §6.1, we specialize to the case where the hybrid model is a so-called geometric phase (recall Definition 1.4.5). We will define a *virtual fundamental class* using the $\mathbb{Z}_2$-localized Chern class of [PV01]. This class can be used to define enumerative invariants via a Chow or cohomology level Fourier–Mukai transform.

The remainder of §6.1.1 is then devoted to showing that these invariants agree with the GLSM invariants of Definition 5.5.1 (See Proposition 6.1.5). This will be used in §6.1.2 to compare our invariants to cosection localized invariants and certain Gromov–Witten invariants.

Specializing to a geometric phase implies $\mathbf{deg} = 1$, therefore $\langle J \rangle = 1$ and $\Gamma = G \times \mathbb{C}^*_R$. The character $\chi \in \widehat{\Gamma}$ is just the projection onto $\mathbb{C}^*_R$, and $\eta_{\mathbf{deg}} = \eta_1 = \mathrm{id}_{\mathbb{C}^*_R}$.

Consider the vector bundle $\mathcal{E}$ (with fiber V_2) on $\mathcal{X}$, whose total space is

$$\mathrm{tot}\, \mathcal{E} = \mathcal{T} := [(V_1^{ss}(\theta) \times V_2)/G],$$

with projection $q\colon \mathcal{T} \to \mathcal{X}$. Since w is linear on V_2, it gives rise to a section $f \in H^0(\mathcal{X}, \mathcal{E}^\vee)$ of the *dual* vector bundle $\mathcal{E}^\vee$. Recall that nondegeneracy of the hybrid model implies that f is a regular section with smooth zero locus $\mathcal{Z} := Z(f) = Z(dw)$. For the remainder of this subsection, we will assume that $\mathcal{X}$ is a *smooth variety.*

The notations in §4 simplify, since $G_1 = G$, $\mathcal{X} = \mathcal{X}^{\mathrm{rig}} = [V_1^{ss}(\theta)/G]$, and

$$LG_{g,r}(\mathcal{X}, d) \cong \overline{\mathcal{M}}_{g,r}(\mathcal{X}, d), \quad LG_{g,r}(\mathcal{Z}, d) \cong \overline{\mathcal{M}}_{g,r}(\mathcal{Z}, d)).$$

For simplicity we will sometimes denote these spaces by $LG(\mathcal{X})$ and $LG(\mathcal{Z})$ respectively.

Let $\mathcal{V}_2'$ denote $\mathcal{V}_2(-\mathscr{G})$. By Proposition 3.5.5, we may choose an admissible resolution $[\bar{A} \to \bar{B}]$ of $\mathbb{R}\pi_*\mathcal{V}$ over U which splits as

$$[\bar{A}_1 \to \bar{B}_1] \oplus [\bar{A}_2' \oplus \mathcal{V}_2|_{\mathscr{G}} \to \bar{B}_2]$$

where

$$[\bar{A}_2' \oplus \mathcal{V}_2|_{\mathscr{G}} \to \bar{B}_2] \cong \mathbb{R}\pi_*\mathcal{V}_2 \text{ and } [\bar{A}_2' \to \bar{B}_2] \cong \mathbb{R}\pi_*\mathcal{V}_2'.$$

Let $\square'$ denote the intersection of $\square$ and $\mathrm{tot}\, \bar{A}_1 \oplus \bar{A}_2'$. Let U' denote the intersection of $\square$ and $\mathrm{tot}\, \bar{A}_1$.

Remark 6.1.1. The space $\square'$ can be constructed from scratch by mimicking the construction of $\square$, but replacing every instance of $\omega_{\mathfrak{C}}^{\log}$ with $\omega_{\mathfrak{C}}$.

Recall from §5.1 that we also have a resolution

$$[\widetilde{A}_1 \to \widetilde{B}_1] \cong \mathbb{R}\pi_*\mathcal{V}_1$$

lying over the closure $\overline{U}$ together with a surjection $\bar{A}_1|_U \twoheadrightarrow \widetilde{A}_1|_U$ and the evaluation map $U' \to \mathcal{X}^r$ factors through

$$U' \to \mathrm{tot}(\bar{A}_1) \to \mathrm{tot}(\widetilde{A}_1) \to [V_1/G]^r.$$

Let $\tilde{d} : \widetilde{U} \to \overline{U}$ denote the desingularization of $\overline{U}$ from §5.5 and define

$$P := \mathrm{Proj}(\mathrm{tot}(\tilde{d}^*(\widetilde{A}_1) \oplus \mathcal{O}_{\widetilde{U}})).$$

The map $\widetilde{A}_1 \to \mathcal{V}_1|_{\mathscr{G}}$ defines a geometric evaluation map $\mathrm{tot}(\widetilde{A}_1) \to [V_1/G]^r$. Let $\mathrm{tot}(\widetilde{A}_1)^\circ$ denote the preimage of the semi-stable locus, so we obtain

$$\mathrm{tot}(\widetilde{A}_1)^\circ \to \mathcal{X}^r.$$

This defines a rational map to $\mathrm{ev}_P : P \dashrightarrow \mathcal{X}^r$. Let $\widetilde{P}$ denote a smooth resolution of $\overline{\Gamma_{\mathrm{ev}_P}} \in P \times \mathcal{X}^r$.

We have the following commuting diagram

$$\begin{tikzcd}
\square \arrow[r, "z_{\mathcal{T}}"] \arrow[d, "l"] & \mathrm{tot}(\bar{A})^\circ \arrow[d, "\bar{l}"] \arrow[dr, "f_{\mathcal{T}}"] & & & \\
\square' \arrow[r, "z'_{\mathcal{T}}"] \arrow[d, "m"] \arrow[u, bend left, "s"] & \mathrm{tot}(\bar{A}_1 \oplus \bar{A}'_2)^\circ \arrow[d] \arrow[dr, "f'_{\mathcal{T}}"] & \mathrm{Tot}^\circ \arrow[r, "i_{\mathcal{T}}"] \arrow[d, "l^\circ"] & \widetilde{\mathrm{Tot}} \arrow[r, "\mathrm{ev}_{\mathcal{T}}"] \arrow[d, "\tilde{l}"] & \mathcal{T}^r \arrow[d] \\
U' \arrow[r, "z_{\mathcal{X}}"] & \mathrm{tot}(\bar{A}_1)^\circ \arrow[r, "f_{\mathcal{X}}"] & \mathrm{tot}(\widetilde{A}_1|_{\widetilde{U}})^\circ \arrow[r, "i_{\mathcal{X}}"] & \widetilde{P} \arrow[r, "\mathrm{ev}_{\mathcal{X}}"] & \mathcal{X}^r \\
LG(\mathcal{Z}) \arrow[u] \arrow[rr, "\iota"] & & P_{\mathcal{Z}}{}^\circ \arrow[u, "k^\circ"] \arrow[r, "j_{\mathcal{Z}}"] \arrow[dr, "i_{\mathcal{Z}}"'] & P_{\mathcal{Z}} \arrow[u, "k"] \arrow[r, "\mathrm{ev}_{\mathcal{Z}}"] & \mathcal{Z}^r \arrow[u] \\
& & & \widetilde{P}_{\mathcal{Z}} \arrow[u, "d_{\mathcal{Z}}"] \arrow[uu, bend right, "\tilde{k}"'] &
\end{tikzcd} \tag{6.1}$$

where the remaining spaces are defined so that all small squares (and the parallelogram) are cartesian (although the bottom left rectangle is commutative), and $d_{\mathcal{Z}} : \widetilde{P}_{\mathcal{Z}} \to P_{\mathcal{Z}}$ is a desingularization of $P_{\mathcal{Z}}$. For convenience we will let n denote the composition $n := i_{\mathcal{T}} \circ f_{\mathcal{T}} \circ z_{\mathcal{T}} : \square \to \widetilde{\mathrm{Tot}}$. Note that the evaluation map $\mathrm{ev} : \square \to \mathcal{T}^r$ factors as $\mathrm{ev}_{\mathcal{T}} \circ n$.

Recall from Definition 2.5.4, there is an equivalence of categories

$$\tilde{\phi}_+ : \mathrm{D}(\mathcal{Z}) \xrightarrow{\sim} \mathrm{D}([\mathcal{T}/\mathbb{C}^*_R], w),$$

which sends the structure sheaf $\mathcal{O}_{\mathcal{Z}}$ to the Koszul factorization S_1 of (2.4).

The fundamental factorization in Definition 5.1.4

$$K = K_{g,r,d} \in \mathrm{D}([\square/\mathbb{C}^*_R], -\mathrm{ev}^* \boxplus_{i=1}^r w)$$

is defined as the Koszul factorization on the section and cosection

$$\mathcal{O}_\square \xrightarrow{\beta} \bar{B} \xrightarrow{-\alpha^\vee} \mathcal{O}_\square$$

i.e.

$$K = \{-\alpha, \beta\}.$$

(Note: to declutter notation, in this section, we use $\bar{A}, \bar{B}$, etc. to denote the corresponding pullback of the bundle to whichever space we are working on.)

The tensor product of K with $n^* \circ \mathrm{ev}_{\mathcal{T}}^*(S_1^{\boxtimes r})$ is the Koszul factorization

$$K^{S_1} := \{\alpha^{S_1}, \beta^{S_1}\} \in \mathrm{D}([\square/\mathbb{C}^*_R], 0)$$

where

$$\alpha^{S_1} := (-\alpha, \mathrm{ev}^*(q^* f)^r) \quad \text{and} \quad \beta^{S_1} := (\beta, \mathrm{ev}^* \mathrm{taut})$$

are a cosection and a section of $\bar{B}^{S_1} := \bar{B} \oplus \pi_*(\mathcal{V}_2|_{\mathscr{G}})$.

Recall that $LG_{g,r}(\mathcal{T}, d) = Z(\beta) \subseteq \square$. Similarly let

$$LG(\mathcal{T}, d)' := Z(\beta|_{\square'}) \subseteq \square'.$$

Furthermore, as $LG_{g,r}(\mathcal{Z}, d)$ lies in $\square'$, we have

$$LG_{g,r}(\mathcal{Z}, d) = Z(\beta) \cap Z(\alpha) \subseteq \square' \subseteq \square$$

by Lemma 5.1.5.

Let K' denote $s^*(K)$. Note that this is a factorization of zero on $\square'$. By [PV16, Proposition 4.3.1], we observe that $\mathbb{R}s_*(K') = K^{S_1}$.

Definition 6.1.2. *Define the virtual fundamental class to be the cycle class,*

$$\begin{aligned}[LG_{g,r}(\mathcal{Z},d)]^{\mathrm{vir}} &:= \mathrm{td}(\bar{B}|_{\square'})\ {}^{\mathbb{Z}_2}\mathrm{ch}^{\square'}_{LG_{g,r}(\mathcal{Z},d)}(K')[\square'] \\ &\in A_*(LG_{g,r}(\mathcal{Z},d))_{\mathbb{Q}}.\end{aligned}$$

Using [Hir17a], we can translate between the Fourier–Mukai transform for a geometric phase using a fundamental factorization and a Fourier–Mukai transform in the traditional sense, using an object of the derived category. Namely we have the following lemma.

Lemma 6.1.3. *There exists an object $K_{\mathcal{Z}} \in \mathrm{D}(\widetilde{P}_{\mathcal{Z}})$ such that*

$$(\Phi_{K_{g,r,d}})_* \circ ((\tilde{\phi}_+)^r)_* = (\Phi_{K_{\mathcal{Z}}})_*, \tag{6.2}$$

where $\Phi_{K_{\mathcal{Z}}} : \mathcal{Z}^r \to \widetilde{U}$ denotes the Fourier–Mukai transform with kernel $K_{\mathcal{Z}}$. Furthermore, $K_{\mathcal{Z}}$ satisfies

$$\mathbb{R}(\tilde{l} \circ n)_* K^{S_1} = \mathbb{R}\tilde{k}_* K_{\mathcal{Z}}. \tag{6.3}$$

Proof. Since $\mathrm{ev}_{\mathcal{X}} \circ i_{\mathcal{X}} : \mathrm{tot}(\widetilde{A}_1|_{\widetilde{U}})^\circ \to \mathcal{X}^r$ is smooth, $(\mathrm{ev}_{\mathcal{X}} \circ i_{\mathcal{X}})^* f$ gives a regular section of $\mathcal{V}_2|_{\mathscr{G}}$. Hence, we can apply Theorem 2.5.2 and Proposition 2.5.3 to obtain an object $K_{\mathcal{Z}}^{\mathrm{pre}} \in \mathrm{D}(\mathrm{tot}(\widetilde{A}_1|_{\widetilde{U}})^\circ)_{LG(\mathcal{Z})}$ such that the following diagram is commutative diagram

$$\begin{tikzcd} \mathrm{D}([\mathrm{Tot}^\circ/\mathbb{C}^*_R], \mathrm{ev}_{\mathcal{T}} \circ i^*_{\mathcal{T}} \boxplus_{i=1}^r w) \arrow[r, "-\overset{\mathbb{L}}{\otimes}\mathbb{R}(z_{\mathcal{T}}\circ f_{\mathcal{T}})_*K"] & \mathrm{D}([\mathrm{Tot}^\circ/\mathbb{C}^*_R], 0)_{[LG(\mathcal{Z})/\mathbb{C}^*_R]} \arrow[d, "\mathbb{R}l^\circ_*"] \\ \mathrm{D}(P^\circ_{\mathcal{Z}}) \arrow[u, "\tilde{\phi}_{P^\circ_{\mathcal{Z}},+}"] \arrow[r, "\mathbb{R}k^\circ_*(-\overset{\mathbb{L}}{\otimes}K^{\mathrm{pre}}_{\mathcal{Z}})"] & \mathrm{D}(\mathrm{tot}(\widetilde{A}_1|_{\widetilde{U}})^\circ)_{LG(\mathcal{Z})} \end{tikzcd} \tag{6.4}$$

where $\tilde{\phi}_{P^\circ_{\mathcal{Z}},+}$ is notation for $\tilde{\phi}_+$ in the particular case $Z = P^\circ_{\mathcal{Z}}$ and $T = \mathrm{Tot}^\circ$.

We can then define

$$K_{\mathcal{Z}} := (i_{\mathcal{Z}})_* K^{\mathrm{pre}}_{\mathcal{Z}} \in \mathrm{D}(\widetilde{P}_{\mathcal{Z}}).$$

Let $\tilde{\phi}_{P_{\mathcal{Z}},+}$ denote $\tilde{\phi}_+$ in the particular case $Z = P_{\mathcal{Z}}$ and $T = \widetilde{\mathrm{Tot}}$ (in this case, it may not be an equivalence but the functor exists nonetheless).

We claim the following diagram, which implies (6.2), commutes.

$$\begin{tikzcd} \mathrm{D}([\mathcal{T}^r/\mathbb{C}^*_R], \boxplus_{i=1}^r w) \arrow[r, "\mathbb{L}\,\mathrm{ev}^*_{\mathcal{T}}"] & \mathrm{D}([\widetilde{\mathrm{Tot}}/\mathbb{C}^*_R], \mathrm{ev}^*_{\mathcal{T}} \boxplus_{i=1}^r w) \arrow[r, "-\overset{\mathbb{L}}{\otimes}\mathbb{R}n_*K"] & \mathrm{D}([\widetilde{\mathrm{Tot}}/\mathbb{C}^*_R], 0)_{[LG(\mathcal{Z})/\mathbb{C}^*_R]} \arrow[d, "\mathbb{R}\tilde{l}_*"] \\ \mathrm{D}(\mathcal{Z}^r) \arrow[u, "(\tilde{\phi}_+)^r"] \arrow[r, "\mathbb{L}(\mathrm{ev}_{\mathcal{Z}})^*"] \arrow[dr, "\mathbb{L}(\mathrm{ev}_{\mathcal{Z}} \circ d_{\mathcal{Z}})^*"'] & \mathrm{D}(P_{\mathcal{Z}}) \arrow[u, "\tilde{\phi}_{P_{\mathcal{Z}},+}"] \arrow[r, "\mathbb{R}k_*(-\overset{\mathbb{L}}{\otimes}\mathbb{R}(d_{\mathcal{Z}})_*K_{\mathcal{Z}})"] & \mathrm{D}(\widetilde{P}) \\ & \mathrm{D}(\widetilde{P}_{\mathcal{Z}}) \arrow[ur, "\mathbb{R}\tilde{k}_*(-\overset{\mathbb{L}}{\otimes}K_{\mathcal{Z}})"'] & \end{tikzcd} \tag{6.5}$$

The left square follows from flat base change. For the right square, we have

$$\begin{aligned}
\mathbb{R}\tilde{l}_*(\tilde{\phi}_{P_{\mathcal{Z}},+}(A) \overset{\mathbb{L}}{\otimes} \mathbb{R}n_*K) &= \mathbb{R}\tilde{l}_*(\tilde{\phi}_{P_{\mathcal{Z}},+}(A) \overset{\mathbb{L}}{\otimes} \mathbb{R}n_*\mathbb{R}(i_{\mathcal{T}})_*\mathbb{L}i^*_{\mathcal{T}}K) \\
&= \mathbb{R}\tilde{l}_*\mathbb{R}(i_{\mathcal{T}})_*(\mathbb{L}i^*_{\mathcal{T}}\tilde{\phi}_{P_{\mathcal{Z}},+}(A) \overset{\mathbb{L}}{\otimes} \mathbb{R}n_*\mathbb{L}i^*_{\mathcal{T}}K) \\
&= \mathbb{R}(i_{\mathcal{X}})_*\mathbb{R}(l^\circ)_*(\phi_{P^\circ_{\mathcal{Z}},+}(\mathbb{L}j^*_{\mathcal{Z}}A) \overset{\mathbb{L}}{\otimes} \mathbb{R}(z_{\mathcal{T}} \circ f_{\mathcal{T}})_*K) \\
&= \mathbb{R}(i_{\mathcal{X}})_*\mathbb{R}k^\circ_*(\mathbb{L}j^*_{\mathcal{Z}}A \overset{\mathbb{L}}{\otimes} K^{\mathrm{pre}}_{\mathcal{Z}}) \\
&= \mathbb{R}k_*\mathbb{R}(j_{\mathcal{Z}})_*(\mathbb{L}j^*_{\mathcal{Z}}A \overset{\mathbb{L}}{\otimes} K^{\mathrm{pre}}_{\mathcal{Z}}) \\
&= \mathbb{R}k_*(A \overset{\mathbb{L}}{\otimes} \mathbb{R}(d_{\mathcal{Z}})_*K_{\mathcal{Z}}).
\end{aligned}$$

Most of these equalities are simply the projection formula of Proposition 2.2.10, however the fourth equality needs justification. This one follows from (6.4). Finally, for the bottom triangle we have

$$\begin{aligned}
\mathbb{R}k_*(\mathbb{L}(\mathrm{ev}_{\mathcal{Z}})^*A \overset{\mathbb{L}}{\otimes} \mathbb{R}(d_{\mathcal{Z}})_*K_{\mathcal{Z}})) &= \mathbb{R}k_*(\mathbb{L}(\mathrm{ev}_{\mathcal{Z}})^*A \overset{\mathbb{L}}{\otimes} \mathbb{R}(j_{\mathcal{Z}})_*K^{\mathrm{pre}}_{\mathcal{Z}}) \\
&= \mathbb{R}k_*\mathbb{R}(j_{\mathcal{Z}})_*(\mathbb{L}j^*_{\mathcal{Z}}\mathbb{L}(\mathrm{ev}_{\mathcal{Z}})^*A \overset{\mathbb{L}}{\otimes} K^{\mathrm{pre}}_{\mathcal{Z}}) \\
&= \mathbb{R}\tilde{k}_*\mathbb{R}(i_{\mathcal{Z}})_*(\mathbb{L}(i_{\mathcal{Z}})^*\mathbb{L}(\mathrm{ev}_{\mathcal{Z}} \circ d_{\mathcal{Z}})^*A \overset{\mathbb{L}}{\otimes} K^{\mathrm{pre}}_{\mathcal{Z}})) \\
&= \mathbb{R}\tilde{k}_*(\mathbb{L}(\mathrm{ev}_{\mathcal{Z}} \circ d_{\mathcal{Z}})^*A \overset{\mathbb{L}}{\otimes} \mathbb{R}(i_{\mathcal{Z}})_*K^{\mathrm{pre}}_{\mathcal{Z}})) \\
&= \mathbb{R}\tilde{k}_*(\mathbb{L}(\mathrm{ev}_{\mathcal{Z}} \circ d_{\mathcal{Z}})^*A \overset{\mathbb{L}}{\otimes} K_{\mathcal{Z}}).
\end{aligned}$$

Now to justify (6.3), plug $\mathcal{O}_{P_{\mathcal{Z}}}$ into the square on the right of the (6.5) to get,

$$\begin{aligned}
\mathbb{R}(\tilde{l} \circ n)_*K^{S_1} &= \mathbb{R}\tilde{l}_* \circ \mathbb{R}n_*(K \overset{\mathbb{L}}{\otimes} \mathbb{L}(\mathrm{ev}_{\mathcal{T}} \circ n)^*(\tilde{\phi}_+)^r\, \mathcal{O}_{\mathcal{Z}^r}) \\
&= \mathbb{R}\tilde{l}_*(\mathbb{R}n_*K \overset{\mathbb{L}}{\otimes} \mathbb{L}(\mathrm{ev}_{\mathcal{T}})^*(\tilde{\phi}_+)^r\, \mathcal{O}_{\mathcal{Z}^r}) \\
&= \mathbb{R}\tilde{l}_*(\mathbb{R}n_*K \overset{\mathbb{L}}{\otimes} \tilde{\phi}_{P_{\mathcal{Z}},+}\, \mathcal{O}_{P_{\mathcal{Z}}}) \\
&= \mathbb{R}k_*\mathbb{R}(d_{\mathcal{Z}})_*K_{\mathcal{Z}} \\
&= \mathbb{R}\tilde{k}_*K_{\mathcal{Z}}.
\end{aligned}$$

□

Lemma 6.1.4. *The following equality holds in* $A_*(LG_{g,r}(\mathcal{Z}, d))_{\mathbb{Q}}$

$$[LG_{g,r}(\mathcal{Z}, d)]^{\mathrm{vir}} = \mathrm{td}(T_{\widetilde{P}_{\mathcal{Z}}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}})\, \mathrm{td}(\mathbb{R}\pi_*\mathcal{V}')^{-1}\, \mathrm{ch}^{\widetilde{P}_{\mathcal{Z}}}_{LG(\mathcal{Z})}(K_{\mathcal{Z}})[\widetilde{P}_{\mathcal{Z}}].$$

Proof. Now, we have the following sequence of equalities

$$\begin{aligned}
[LG_{g,r}(\mathcal{Z}, d)]^{\mathrm{vir}} &= \mathrm{td}(\bar{B}|_{\square'})\ {}^{\mathbb{Z}_2}\mathrm{ch}^{\square'}_{LG(\mathcal{Z})}(K')[\square'] \\
&= \mathrm{td}(\bar{B})\, \mathrm{td}(\bar{A}_2{}')^{-1}\, \mathrm{ch}^{U'}_{LG(\mathcal{Z})}(\mathbb{R}(m \circ l)_*K^{S_1})[U'] \\
&= \mathrm{td}(\bar{B} \oplus \widetilde{A}_1 \oplus A_1)\, \mathrm{td}(\bar{A}_1 \oplus \bar{A}_2{}')^{-1}\, \mathrm{ch}^{\mathrm{tot}\, \widetilde{A}_1|_{\widetilde{U}}}_{LG(\mathcal{Z})}(\mathbb{R}(l^\circ \circ f_{\mathcal{T}} \circ z_{\mathcal{T}})_*K^{S_1})[\mathrm{tot}\, \widetilde{A}_1|_{\widetilde{U}}] \\
&= \mathrm{td}(\bar{B} \oplus \widetilde{A}_1 \oplus A_1)\, \mathrm{td}(\bar{A}_1 \oplus \bar{A}_2{}')^{-1}\, \mathrm{ch}^{\widetilde{P}}_{LG(\mathcal{Z})}(\mathbb{R}(\tilde{l} \circ n)_*K^{S_1})[\widetilde{P}] \\
&= \mathrm{td}(T_{\widetilde{P}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}})\, \mathrm{td}(\mathbb{R}\pi_*\mathcal{V}')^{-1}\, \mathrm{ch}^{\widetilde{P}}_{LG(\mathcal{Z})}(\mathbb{R}(\tilde{l} \circ n)_*K^{S_1})[\widetilde{P}]
\end{aligned}$$

$$= \operatorname{td}(T_{\widetilde{P}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}}) \operatorname{td}(\mathbb{R}\pi_*\mathcal{V}')^{-1} \operatorname{ch}^{\widetilde{P}}_{LG(\mathcal{Z})}(\mathbb{R}\tilde{k}_* K_{\mathcal{Z}})[\widetilde{P}]$$
$$= \operatorname{td}(T_{\widetilde{P}_{\mathcal{Z}}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}}) \operatorname{td}(\mathbb{R}\pi_*\mathcal{V}')^{-1} \operatorname{ch}^{\widetilde{P}_{\mathcal{Z}}}_{LG(\mathcal{Z})}(K_{\mathcal{Z}})[\widetilde{P}_{\mathcal{Z}}] \tag{6.6}$$

where $\mathcal{V}'$ denotes $\mathcal{V}_1 \oplus \mathcal{V}_2(-\mathscr{G})$. The first equality is the definition. The second comes from Proposition 2.5.8, note that we are implicitly identifying $\mathrm{D}(U',0)$ with $\mathrm{D}(U')$. The third follows from [Ful13, Corollary 18.1.2]. The fourth equality is obvious from the definition of localized Chern characters. The fifth line comes from the equalities,

$$T_{\widetilde{P}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}} = A_1 \oplus \widetilde{A}_1 \quad \text{and} \quad \mathbb{R}\pi_*\mathcal{V}' = \bar{A}_1 \oplus \bar{A}_2' \oplus \bar{B}[1].$$

The sixth equality follows from (6.3). The seventh follows from [Ful13, Theorem 18.2] (using the isomorphism $K_0(LG(\mathcal{Z})) \cong K_0(\mathrm{D}(\widetilde{P})_{LG(\mathcal{Z})})$. □

We now prove that our GLSM invariants are the same as those defined by $[LG_{g,r}(\mathcal{Z},d)]^{\mathrm{vir}}$. In the next subsection, we will argue that these agree with the co-section localized Gromov–Witten type invariants defined by Chang–Li [CL11] and and generalized by Fan–Jarvis–Ruan [FJR17].

By abuse of notation we will let $\iota_*[LG_{g,r}(\mathcal{Z},d)]^{\mathrm{vir}}$ and $\operatorname{ch}(K_{\mathcal{Z}})$ now denote the corresponding classes in cohomology rather than the Chow ring.

Proposition 6.1.5. *Let the setup be as in the beginning of this section. Given $\vec{\gamma} = (\gamma_1, \dots, \gamma_r) \in \mathrm{H}^*(\mathcal{Z})^{\otimes r}$, the following are equal:*

$$(\mathrm{proj}_* \circ \tilde{k})_*(\mathrm{ev}^*_{\mathcal{Z}}\, \vec{\gamma} \cup (i_{\mathcal{Z}} \circ \iota)_*[LG_{g,r}(\mathcal{Z},d)]^{\mathrm{vir}}) = \Lambda_{g,r,d}(\varphi^{\mathrm{td}}_*\gamma_1, \dots, \varphi^{\mathrm{td}}_*\gamma_r).$$

Proof. Let $\tilde{u}$ denote the map $\tilde{u} : \widetilde{P} \to \widetilde{U}$. We have,

$$\begin{aligned}
&\tilde{u}_* \circ \tilde{k}_*(\bar{\mathrm{ev}}^*\vec{\gamma} \cup (i_{\mathcal{Z}} \circ \iota)_*[LG_{g,r}(\mathcal{Z},d)]^{\mathrm{vir}}) \\
=&\tilde{u}_* \circ \tilde{k}_*(\operatorname{td}(T_{\widetilde{P}_{\mathcal{Z}}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}}) \operatorname{td}(\mathbb{R}\pi_*\mathcal{V}')^{-1} \operatorname{ch}(K_{\mathcal{Z}}) ev^*\vec{\gamma}) \\
=& \operatorname{td}(T_{\widetilde{U}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}}) \operatorname{td}(\mathbb{R}\pi_*\mathcal{V}')^{-1} \tilde{u}_* \circ \tilde{k}_*(\operatorname{td}(T_{\widetilde{P}_{\mathcal{Z}}/\widetilde{U}}) \operatorname{ch}(K_{\mathcal{Z}}) ev^*\vec{\gamma}) \\
=& \operatorname{td}(T_{\widetilde{U}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}}) \operatorname{td}(\mathbb{R}\pi_*\mathcal{V}')^{-1} (\Phi^{\mathrm{H}}_{\operatorname{td}(T_{\widetilde{P}_{\mathcal{Z}}/\widetilde{U}}) \operatorname{ch}(K_{\mathcal{Z}})})_*(\vec{\gamma}) \\
=& \operatorname{td}(T_{\widetilde{U}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}}) \operatorname{td}(\mathbb{R}\pi_*\mathcal{V}')^{-1} \bar{\phi}_{\mathrm{HKR}} (\Phi_{K_{\mathcal{Z}}})_* \phi^{-1}_{\mathrm{HKR}}(\vec{\gamma}) \\
=& \operatorname{td}(T_{\widetilde{U}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}}) \operatorname{td}(\mathbb{R}\pi_*\mathcal{V}')^{-1} \bar{\phi}_{\mathrm{HKR}} (\Phi_{K_{g,r,d}})_*((\tilde{\phi}_+)_* \circ \phi^{-1}_{\mathrm{HKR}}\gamma_1, \dots, (\tilde{\phi}_+)_* \circ \phi^{-1}_{\mathrm{HKR}}\gamma_r) \\
=& \operatorname{td}(T_{\widetilde{U}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}}) \operatorname{td}(\mathbb{R}\pi_*\mathcal{V})^{-1} \bar{\phi}_{\mathrm{HKR}} (\Phi_{K_{g,r,d}})_*(\varphi^{\mathrm{td}}_*\gamma_1, \dots, \varphi^{\mathrm{td}}_*\gamma_r).
\end{aligned}$$

The first equality is by Lemma 6.1.4. The second is the projection formula. The third is just the definition of the cohomological Fourier–Mukai transform and the fourth equality is Theorem 2.5.1. The fifth equality is by (6.2). The last equality follows from $\mathbb{R}\pi_*\mathcal{V} = \mathbb{R}\pi_*\mathcal{V}' \oplus \pi_*\mathcal{V}_2|_{\mathscr{G}}$ and the definition of φ^{td}_* (Definition 2.5.7). Note that the maps ord and q^* of Definition 5.5.1 are both the identity in this case, since $\mathcal{X}$ is a smooth variety. Applying proj_* finishes the theorem. □

6.1.2. *Cosection Localized Gromov–Witten invariants.* In [CL11], the cosection localization methods of Kiem–Li [KL13] are used to construct a cosection localized virtual class for a particular geometric phase GLSM, namely the quintic 3-fold GLSM of Example 1.4.7. The cosection localized virtual class is used to define enumerative invariants for general GLSMs in [FJR17]. The construction can be summarized as follows (see Section 3 of [CL11] and Section 5 of [FJR17] for details).

The object

$$\mathbb{E} := \mathbb{R}\pi_*\mathcal{V}$$

gives a perfect obstruction theory over $LG(\mathcal{T})'$ relative to $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, d)^{\circ}_{\omega^{\log}_{\mathfrak{C}}}$. Then given a potential $w : \mathcal{T} \to \mathbb{A}^1$, Equation (3.29) defines a cosection dw of $\mathcal{O}b_{LG(\mathcal{T})'} = \mathbb{R}^1\pi_*\mathcal{V}(-\mathscr{G})$. Let $h^1/h^0(\mathbb{E})(dw)$ be the refined cone stack associated to $\mathbb{E}$ and dw, (see Equation (3.16) of [CL11]). As shown in the proof of Proposition 6.1.7, the relative intrinsic normal cone $[c_{LG(\mathcal{T})'/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)^{\circ}_{\omega^{\log}_{\mathfrak{C}}}}]$ lies in $Z_*(h^1/h^0(\mathbb{E})(dw))_{\mathbb{Q}}$. Hence, we may define the following class.

Definition 6.1.6. *The cosection localized virtual fundamental class is the class*

$$[LG_{g,r}(\mathcal{T}, d)']^{\mathrm{vir}}_{dw} := s^!_{h^1/h^0(\mathbb{E}),dw}[c_{LG(\mathcal{T})'/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)^{\circ}_{\omega^{\log}_{\mathfrak{C}}}}].$$

where

$$s^!_{h^1/h^0(\mathbb{E}),dw} : A_*(h^1/h^0(\mathbb{E})(dw))_{\mathbb{Q}} \to A_*([LG_{g,r}(\mathcal{Z}, d)])_{\mathbb{Q}}$$

is the cosection localized Gysin map [KL13, Proposition 1.3].

The following result follows from results in [CLL15] and the compatibility of α and dw (Lemma 3.6.3).

Proposition 6.1.7. *The following equality holds in* $A_*(LG_{g,r}(\mathcal{Z}, d))_{\mathbb{Q}}$

$$[LG_{g,r}(\mathcal{Z}, d)]^{\mathrm{vir}} = [LG_{g,r}(\mathcal{T}, d)']^{\mathrm{vir}}_{dw}$$

Proof. Note that $LG(\mathcal{T})'$ is realized as the zero locus of a section $\beta \in \Gamma(\square', \bar{B})$ in the smooth Deligne–Mumford stack $\square'$, and that over $\square'$, $\mathbb{R}\pi_*\mathcal{V}$ is resolved by the two term complex $[\bar{A}_1 \oplus \bar{A}'_2 \to \bar{B}]$. In this special setting, the cosection localized virtual class can be described more concretely.

Let $C_{LG(\mathcal{T})'/\square'}$ be the normal cone to $LG(\mathcal{T})'$ in $\square'$. Since $LG(\mathcal{T})' = Z(\beta)$, the cone $C_{LG(\mathcal{T})'/\square'}$ can be viewed as a closed substack in $\bar{B}|_{LG(\mathcal{T})'}$. Since $\alpha|_{LG(\mathcal{T})'} \in \mathrm{Hom}(\bar{B}|_{LG(\mathcal{T})'}, \mathcal{O}_{LG(\mathcal{T})'})$ becomes zero when it is restricted to $C_{LG(\mathcal{T})'/\square'}$ by $\alpha \circ \beta = 0$ in $\square'$, the cone cycle is contained in

$$\bar{B}|_{LG(\mathcal{T})'}(\alpha|_{LG(\mathcal{T})'}) := \bar{B}|_{Z(\alpha,\beta)} \cup \ker(\bar{B}|_{LG(\mathcal{T})' \setminus Z(\alpha,\beta)} \to \mathcal{O}_{LG(\mathcal{T})' \setminus Z(\alpha,\beta)}).$$

Again there exists a cosection localized Gysin map

$$s^!_{\bar{B}|_{LG(\mathcal{T})'},\alpha|_{LG(\mathcal{T})'}} : A_*(\bar{B}|_{LG(\mathcal{T})'}(\alpha|_{LG(\mathcal{T})'}))_{\mathbb{Q}} \to A_*(Z(\alpha, \beta))_{\mathbb{Q}}.$$

The relative instrinsic normal cone $c_{LG(\mathcal{T})'/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)^{\circ}_{\omega^{\log}_{\mathfrak{C}}}}$ can be explicitly described as the quotient cone stack $[C_{LG(\mathcal{T})'/\square'}/\bar{A}]$ since $\square'$ is smooth with relative tangent bundle $\bar{A}$. This yields the following fiber diagram

$$\begin{tikzcd} C_{LG(\mathcal{T})'/\square'} \arrow[r] \arrow[d] & \operatorname{tot}\bar{B} \arrow[d] \\ c_{LG(\mathcal{T})'/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)^{\circ}_{\omega^{\log}_{\mathfrak{C}}}} \arrow[r] & [\operatorname{tot}\bar{B}/\bar{A}] \end{tikzcd} \tag{6.7}$$

We now have the following chain of equalities

$$\begin{aligned} [LG_{g,r}(\mathcal{T},d)']^{\mathrm{vir}}_{dw} &= s^{!}_{h^1/h^0(\mathbb{E}),dw}[c_{LG(\mathcal{T})'/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)^{\circ}_{\omega^{\log}_{\mathfrak{C}}}}] \\ &= s^{!}_{\bar{B}|_{LG(\mathcal{T})'},\alpha|_{LG(\mathcal{T})'}}\gamma^*[c_{LG(\mathcal{T})'/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)^{\circ}_{\omega^{\log}_{\mathfrak{C}}}}] \\ &= s^{!}_{\bar{B}|_{LG(\mathcal{T})'},\alpha|_{LG(\mathcal{T})'}}[C_{LG(\mathcal{T})'/\square'}] \\ &= \operatorname{td}(\bar{B}|_{\square'})\ {}^{\mathbb{Z}_2}\mathrm{ch}^{\square'}_{LG_{g,r}(\mathcal{Z},d)}(K')[\square'] \\ &= [LG_{g,r}(\mathcal{Z},d)]^{\mathrm{vir}}. \end{aligned}$$

The first line is by definition. The second line uses the definition of the Gysin map together with the fact that σ agrees with $\alpha|_{LG(\mathcal{T})'}$ (Lemma 3.6.3). The fourth line follows from (6.7). The fifth line is [CLL15, Proposition 5.10]. The final line is also by definition. □

With the above result and Proposition 6.1.5, we can identify our GLSM invariants in the geometric phase with invariants defined by the cosection localized virtual class. In special cases this recovers Gromov–Witten theory.

Theorem 6.1.8. *After identifying* $\mathrm{H}^*(\mathcal{Z})$ *with* $\mathrm{HH}_*([\mathcal{T}/\mathbb{C}^*_R],w)$ *via* φ^{td}_**, the invariants defined by the cosection localized virtual fundamental class of* [CL11] *are equal to the invariants* $\Lambda_{g,r,d}$ *for the corresponding GLSM from Definition 5.5.1.*

Proof. This follows immediately from Propositions 6.1.5 and 6.1.7. □

In some special cases, it is known that cosection localized Gromov–Witten type invariants agree with classical Gromov–Witten type invariants.

Theorem 6.1.9 (Chang–Li)**.** *In the special case of the quintic three-fold GLSM (Example 1.4.7) or more generally a geometric phase with degeneracy locus* $\mathcal{Z}$ *a Calabi–Yau complete intersection three-fold, the genus-*g *degree-*d *(*$r = 0$*) Gromov–Witten invariant of* $\mathcal{Z}$*, denoted* $N^d_g(\mathcal{Z})$*, is equal to the degree of the virtual fundamental class up to a sign:*

$$\deg\left([LG_{g,0}(\mathcal{T},d)]^{\mathrm{vir}}_{dw}\right) = (-1)^{\chi(\mathbb{R}\pi_*(\mathcal{V}_2'))}N^d_g(\mathcal{Z}).$$

Proof. This is [CL11, Theorem 1.1]. We note that although the results in [CL11] are stated only for the quintic threefold, the proof extends to the cases mentioned (this is well known and mentioned in the introduction of [CL11]). □

Corollary 6.1.10. *When* $\mathcal{Z}$ *is a Calabi–Yau complete intersection three-fold, the Gromov–Witten invariant* $N^d_g(\mathcal{Z})$ *is equal to* $\Lambda_{g,r,d}$ *up to a sign.*

Proof. This follows immediately from Theorem 6.1.8 and Theorem 6.1.9. □

Corollary 6.1.11. *In the special case where $V_2 = 0$ and $w = 0$, the Gromov–Witten invariants defined by the Behrend–Fantechi virtual cycle for the GIT quotient $[V_1 /\!\!/_\theta G]$ agree with $\Lambda_{g,r,d}$.*

Proof. In this special case, we get the Behrend–Fantechi virtual cycle without having to apply the cosection localization method i.e. [CLL15, Proposition 5.10] is used in the case where the cosection is 0. □

Remark 6.1.12. We expect more generally that our geometric phase GLSM invariants agree with the usual Gromov–Witten invariants defined in [BF97, LT98] up to sign.

6.2. **Comparison with the Polishchuk–Vaintrob construction.** Let G denote a finite abelian subgroup of $(\mathbb{C}^*)^m$, and let $w : \mathbb{A}^m \to \mathbb{A}^1$ denote a G-invariant function with an isolated singularity at the origin. Suppose that a $\mathbb{C}^*_R$-action on $\mathbb{A}^m$ is given by an embedding into $(\mathbb{C}^*)^m$ such that: w has a homogeneous degree $\mathbf{deg}$ with respect to $\mathbb{C}^*_R$-action and $G \cap \mathbb{C}^*_R$ is a finite cyclic group $\langle J \rangle$ with order $\mathbf{deg}$. With this input data, a cohomological field theory called Fan–Jarvis–Ruan–Witten (FJRW) theory has been developed by Fan–Jarvis–Ruan [FJR13] in the analytic category and by Polishchuk–Vaintrob [PV16] in the algebraic category using factorizations. These theories can be thought of as giving invariants of the singularity defined by

$$w : [\mathbb{A}^m / G] \to \mathbb{A}^1 .$$

A GLSM corresponding to such a singularity is known as an (abelian) affine phase GLSM. We construct it as follows. Take $\Gamma = G \cdot \mathbb{C}^*_R \subseteq (\mathbb{C}^*)^m$, $V_1 = \mathrm{Spec}(\mathbb{C})$, $V_2 = \mathbb{A}^m$, and let $\theta : G \to \mathbb{C}^*$ be trivial. Note that $(V, G, \mathbb{C}^*_R, \theta, w)$ is an abelian GLSM such that $[V_1 /\!\!/_\theta G]$ is the étale gerbe BG. Here χ is defined by $g \cdot \lambda \mapsto \lambda^{\mathbf{deg}}$ for $g \in G, \lambda \in \mathbb{C}^*_R$. In this section we show the invariants $\Lambda_{g,r,d}$ defined for the GLSM $(V, G, \mathbb{C}^*_R, \theta, w)$ agree with Polishchuk–Vaintrob's definition of the FJRW invariants of $w : [\mathbb{A}^m / G] \to \mathbb{A}^1$.

Note first that in the special case where V_1 is rank zero, the stack $LG_{g,r}(BG, 0)$ is a smooth and proper Deligne–Mumford stack over $\mathrm{Spec}(\mathbb{C})$, with projective coarse moduli. This follows from Proposition 4.3.2 after observing that $\overline{\mathcal{M}}_{g,r}(\mathcal{X}^{\mathrm{rig}}, 0) = \overline{\mathcal{M}}_{g,r}(BG_1, 0)$ is smooth.

6.2.1. *The Polishchuk–Vaintrob construction.* In [PV16] the stack $LG_{g,r}(BG, 0)$ is denoted as $\mathcal{S}_{g,r} = \mathcal{S}_{g,r,\Gamma,\chi}$ and is referred to as the moduli of Γ-*spin structures.* We will adopt this notation for the remainder of this section ot simplify the comparison. In [PV16], a cohomological field theory *with coefficients in* $\mathbb{C}[\widehat{G}]$ is defined as follows. First, a rigidified stack $\mathcal{S}^{\mathrm{rig}}_{g,r} \to \mathcal{S}_{g,r}$ is defined, which parametrizes Γ-spin structures together with trivializations

$$\mathcal{P}|_{p_i} \cong \Gamma / \langle h_i \rangle$$

for each marked point $p_1, \ldots, p_r$, where h_i denotes the generator of the isotropy group at p_i, viewed as an element of Γ. There is a natural action of G^r on $S^{\mathrm{rig}}_{g,r}$ by scaling the respective trivializations. Note that the induced action of the diagonal $G \leq G^r$ is trivial.

A $\mathbb{C}^*_R$-equivariant resolution

$$[A \xrightarrow{d} B] \cong \mathbb{R}\pi_*(\mathcal{V})$$

is constructed over $\mathcal{S}^{\mathrm{rig}}_{g,r}$ satisfying Conditions 1, 2, and 3. This is then used to define a Koszul factorization $K^{PV} = K^{PV}_{g,r} \in \mathrm{D}([\mathrm{tot}(A)/\mathbb{C}^*_R], -\mathrm{ev}^*(\boxplus_{i=1}^r w))$.[3]

The G^r action on $S^{\mathrm{rig}}_{g,r}$ extends to an action on $\mathrm{tot}(A)$. The trivializations at the marked points allow one to define a G^r-equivariant evaluation map

$$\mathrm{ev} : \mathrm{tot}(A) \to \coprod_{(\underline{g})} \bigoplus_{i=1}^r V^{g_i},$$

where the disjoint union is over all conjugacy classes (i.e. r-tuples) $(\underline{g})$ in G^r. This defines a functor

$$\begin{array}{rccl} \Phi_{K^{PV}_{g,r}} : & \mathrm{D}(\coprod_{(\underline{g})}[\bigoplus_{i=1}^r V^{g_i}/\Gamma], \boxplus_{i=1}^r w) & \to & \mathrm{D}([S^{\mathrm{rig}}_{g,r}/\Gamma], 0) \cong \mathrm{D}_G(S^{\mathrm{rig}}_{g,r}) \\ & E & \mapsto & p_*(K^{PV}_{g,r} \otimes \mathrm{ev}^*(E)), \end{array}$$

where p is the map $[\mathrm{tot}(A)/\Gamma] \to [\mathcal{S}^{\mathrm{rig}}_{g,r}/\Gamma]$.

It is shown in [PV16, Corollary 2.6.2] that $(\mathcal{H}^{\mathrm{ext}})^{\otimes r}$, the r-fold product of the state space, is isomorphic to $\mathrm{HH}_*(\coprod_{(\underline{g})} \bigoplus_{i=1}^r V^{g_i}, \boxplus_{i=1}^r w)^{G^r}$. We obtain a map on $(\mathcal{H}^{\mathrm{ext}})^{\otimes r}$ as follows:

$$\phi_g : (\mathcal{H}^{\mathrm{ext}})^{\otimes r} \hookrightarrow \mathrm{HH}_* \left(\coprod_{(\underline{g})}[\bigoplus_{i=1}^r V^{g_i}/\Gamma], \boxplus_{i=1}^r w\right) \xrightarrow{\Phi_{K^{PV}_{g,r}\,*}} \mathrm{HH}_*(\mathcal{S}^{\mathrm{rig}}_{g,r} \times BG) \cong$$
$$\mathrm{HH}_*(S^{\mathrm{rig}}_{g,r}) \otimes \mathbb{C}[\widehat{G}] \xrightarrow{\bar{\phi}_{\mathrm{HKR}}} \mathrm{H}^*(\mathcal{S}^{\mathrm{rig}}_{g,r}) \otimes \mathbb{C}[\widehat{G}].$$

Finally, if we let $\mathrm{st}_g : \mathcal{S}^{\mathrm{rig}}_{g,r} \to \overline{\mathcal{M}}_{g,r}$ denote the forgetful map, the Polishchuk–Vaintrob invariants are defined to be

$$\Lambda^{PV}_{g,r} := \frac{1}{\deg(\mathrm{st}_g)} \cdot \mathrm{st}_{g*} \circ \phi_g : (\mathcal{H}^{\mathrm{ext}})^r \to \mathrm{H}^*(\overline{\mathcal{M}}_{g,r}) \otimes \mathbb{C}[\widehat{G}].$$

By [PV16, Theorem 5.1.2], these invariants define a cohomological field theory with coefficients in $\mathbb{C}[\widehat{G}]$.

By [PV16, Theorem 2.6.1] , the state space $\mathcal{H}^{\mathrm{ext}}$ decomposes into a direct sum $\oplus_{h\in G} e_h(\mathcal{H}^{\mathrm{ext}})$ indexed by elements of G (see §5.4 and Remark 5.4.4). More explicitly, $\mathbb{C}[\widehat{G}]$ has an idempotent basis $\{e_h\}_{h\in G}$ where

$$e_h := \frac{1}{|G|} \sum_{\eta \in \widehat{G}} \eta^{-1}(h)[\eta].$$

Here the notation $e_h(\mathcal{H}^{\mathrm{ext}})$ denotes the image of the map e_h in $\mathcal{H}^{\mathrm{ext}}$. Recall that $\mathcal{H}^{\mathrm{red}} := e_{id}(\mathcal{H}^{\mathrm{ext}})$.

Define the *reduced* map by

$$\phi^{\mathrm{red}}_g = \pi_1 \circ \phi_g|_{\mathcal{H}^{\mathrm{red}\otimes r}}$$

where π_1 is the map

$$\mathrm{H}^*(\mathcal{S}^{\mathrm{rig}}_{g,r}) \otimes \mathbb{C}[\widehat{G}] \xrightarrow{e_{id}} \mathrm{H}^*(\mathcal{S}^{\mathrm{rig}}_{g,r}) \otimes \mathbb{C}[e_{id}] \cong \mathrm{H}^*(\mathcal{S}^{\mathrm{rig}}_{g,r}).$$

Let $(\underline{g})$ be an r-tuple of conjugacy classes $(g_1), \ldots, (g_r)$ and choose $s_i \in \mathcal{H}^{\mathrm{red}}_{(g_i)}$. Polishchuk–Vaintrob define the reduced PV invariants by

$$\lambda^{PV}_{g,r}(s_1, \ldots, s_r) := \sigma_g(\underline{g}) \cdot \frac{1}{\deg(\mathrm{st}_g)} \cdot \mathrm{st}_{g*}(\mathrm{td}(R\pi_*(\mathcal{V}_2))^{-1} \cdot \phi^{\mathrm{red}}_g(s_1, \ldots, s_r)),$$

[3]To be precise the resolution is constructed over a different rigidification $S^{\mathrm{rig},\circ}_{g,r}$. The corresponding factorization is then pulled back to define K^{PV} (see the end of §4.2 of [PV16]).

where $\sigma_g(\underline{\boldsymbol{g}})$ is a certain root of unity depending on $(\underline{\boldsymbol{g}})$.

6.2.2. *The comparison.* In this section we prove that the reduced PV invariants λ_g^{PV} agree with those of the GLSM defined in 5.5.1.

Lemma 6.2.1. *The map ϕ_g^{red} is equal to $s^* \circ \phi_g|_{\mathcal{H}^{\mathrm{red}\otimes r}}$ where s is the map*

$$s : \mathcal{S}_{g,r}^{\mathrm{rig}} \to \mathcal{S}_{g,r}^{\mathrm{rig}} \times BG.$$

Proof. The definition of ϕ_g implicitly uses the fact that the vector space $\mathbb{C}[\widehat{G}]$ is isomorphic to $\mathrm{HH}_*(BG) \cong \mathbb{C}[G]$ (see, e.g., [Căl03, Example 6.4]). This isomorphism is given by the map

$$\mathrm{ch} : \mathbb{C}[\widehat{G}] \cong K(BG) \otimes \mathbb{C} \to \mathrm{HH}_*(BG) \cong \mathbb{C}[G]$$

which maps the character $[\eta] \in \widehat{G}$ to

$$\mathrm{ch}([\eta]) := \frac{1}{|G|} \sum_{h \in G} \eta(h^{-1})h.$$

Note that $\mathrm{ch}(e_h) = h/|G|$. Furthermore, under this isomorphism, the decomposition

$$\mathcal{H}^{\mathrm{ext}} = \bigoplus_{h \in G} e_h(\mathcal{H}^{\mathrm{ext}})$$

is compatible with the isomorphism

$$\mathrm{HH}_*(S_{g,r}^{\mathrm{rig}} \times BG) \cong \mathrm{HH}_*(S_{g,r}^{\mathrm{rig}}) \otimes_k \mathrm{HH}_*(BG).$$

Hence, it's enough to consider the case where $S_{g,r}^{\mathrm{rig}}$ is a point.

Consider the map $\bar{s} : pt \to BG$. One observes that the pullback map

$$\bar{s}^* : \mathbb{C}[G] \cong \mathrm{HH}_*(BG) \to \mathrm{HH}_*(pt) \cong \mathbb{C}$$

is given by

$$\bar{s}^*(\mathrm{ch}(e_h)) = \mathrm{ch}(\bar{s}^*(e_h)) = \frac{1}{|G|} \sum_{\eta \in \widehat{G}} \eta^{-1}(h) \, \mathrm{ch}(\mathcal{O}_{pt}) = \delta_{h,id}.$$

Thus applying the operator e_{id} to $\mathbb{C}[\widehat{G}]$ is equivalent to pulling back by $\bar{s}$. □

Let $(V, G, \mathbb{C}_R^*, \theta, w)$ be the GLSM described at the beginning of this section. Note in particular that G is finite.

Proposition 6.2.2. *Over $LG_{g,r}(BG, 0)$ there exists a resolution $[A_0 \xrightarrow{d_0} B_0]$ of $\pi_*(\mathcal{V})$ satisfying Conditions 1, 2, and 3. The induced Koszul factorization $\{-\alpha_0, \beta_0\}$ on $\mathrm{tot}(A_0)$ defines $K_{g,r,0}$.*

In particular, $\Lambda_{g,r,d}(s_1, \dots, s_r)$ can be computed as

$$\Lambda_{g,r,d}(s_1, \dots, s_r) = \mathrm{proj}_* \left(\mathrm{td}(-\mathbb{R}\pi_*\mathcal{V}) \cup \left(\bar{\phi}_{\mathrm{HKR}} \circ \mathrm{ord} \circ q^* \circ (\Phi_{\{-\alpha_0, \beta_0\}})_*(s_1, \dots, s_r)\right)\right)$$

where

$$q : \mathcal{S}_{g,r} \to [\mathcal{S}_{g,r}/\mu_{\mathbf{deg}}]$$

denotes quotient by the trivial action of $\mu_{\mathbf{deg}}$ on $\mathcal{S}_{g,r}$ and $(\Phi_{\{-\alpha_0,\beta_0\}})_* : {\mathcal{H}^{\mathrm{red}}}^{\otimes r} \to \mathcal{S}_{g,r}$ is the integral transform with kernel $\{-\alpha_0, \beta_0\}$.

Proof. Since G is finite, the open subset $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,0)^{\circ}_{\omega^{\log}_{\mathfrak{C}}} \subseteq \mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,0)_{\omega^{\log}_{\mathfrak{C}}}$ is in fact equal to $\mathcal{S}_{g,r} = LG_{g,r}(BG,0)$. In particular it is a quotient stack with projective coarse moduli. This allows the construction of a resolution $[A_0 \xrightarrow{d_0} B_0]$ of $\pi_*(\mathcal{V})$ satisfying Conditions 1, 2, and 3 over $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)^{\circ}_{\omega^{\log}_{\mathfrak{C}}}$. Thus our two-step procedure for constructing the fundamental factorization K is unnecessary; one can use $[A_0 \to B_0]$ to construct a Koszul resolution $\{-\alpha_0,\beta_0\}$ on $\mathrm{tot}(A_0)$. This may be viewed simply as a degenerate case of our general construction, where $V_1 = 0$. □

From [PV16, Section 4], it is apparent that the $\mathbb{C}^*_R$-equivariant resolution $[A \xrightarrow{d} B]$ can be assumed to be the pullback of $[A_0 \xrightarrow{d_0} B_0]$ under the map $\mathcal{S}^{\mathrm{rig}}_{g,r} \to \mathcal{S}_{g,r}$. Consequently the fundamental factorization K^{PV} is the pullback of $\{-\alpha_0,\beta_0\}$ under the map $\mathrm{tot}(A) \to \mathrm{tot}(A_0)$. Indeed this is why $\{-\alpha_0,\beta_0\}$ is referred to as a *PV factorization* (Definition 3.3.4).

Theorem 6.2.3. *The GLSM invariants $\Lambda_{g,r,0}$ are related to the reduced invariants of* [PV16] *by an explicit factor. Given $s_i \in \mathcal{H}^{\mathrm{red}}_{(g_i)}$ for $1 \le i \le r$, let m_i be the order of g_i as in §5.5, then*

$$\begin{aligned}\Lambda_{g,r,0}(s_1,\dots,s_r) &= \tfrac{(m_1\cdots m_r)\deg(\mathrm{proj})}{\sigma_g(\underline{\boldsymbol{g}})}\lambda^{PV}_{g,r}(s_1,\dots,s_r)\\ &= \tfrac{(m_1\cdots m_r)}{\deg(\mathrm{rig})}\mathrm{st}_{g*}(\mathrm{td}(\mathbb{R}\pi_*(\mathcal{V}_2))^{-1}\cdot\phi^{\mathrm{red}}_g(s_1,\dots,s_r)),\end{aligned}$$

where proj *is the forgetful map $\mathcal{S}_{g,r} \to \overline{\mathcal{M}}_{g,r}$.*

Proof. Consider the following commutative diagram.

$$\begin{tikzcd} \mathrm{D}([\mathrm{tot}(A)/\Gamma],0) \arrow[r, "\mathbb{L}\tilde{t}^*"'] \arrow[d, "\mathbb{R}\bar{\pi}_*"] & \mathrm{D}([\mathrm{tot}(A)/\mathbb{C}^*_R],0) \arrow[d, "\mathbb{R}\pi_*"] & \mathrm{D}([\mathrm{tot}(A_0)/\mathbb{C}^*_R],0) \arrow[l, "\mathbb{L}\widetilde{\mathrm{rig}}^*"] \arrow[d, "\mathbb{R}\pi_{0*}"] \\ \mathrm{D}([\mathcal{S}^{\mathrm{rig}}_{g,r}/\Gamma],0) \arrow[r, "\mathbb{L}t^*"'] \arrow[d, "\cong"] & \mathrm{D}([\mathcal{S}^{\mathrm{rig}}_{g,r}/\mathbb{C}^*_R],0) \arrow[d, "\cong"] & \mathrm{D}([\mathcal{S}_{g,r}]/\mathbb{C}^*_R],0) \arrow[l, "\mathbb{L}\overline{\mathrm{rig}}^*"] \arrow[d, "\cong"] \\ \mathrm{D}([\mathcal{S}^{\mathrm{rig}}_{g,r}/G]) \arrow[r, "\mathbb{L}t^*"'] & \mathrm{D}([\mathcal{S}^{\mathrm{rig}}_{g,r}/\mu_{\mathbf{deg}}]) & \mathrm{D}([\mathcal{S}_{g,r}/\mu_{\mathbf{deg}}]) \arrow[l, "\mathbb{L}\overline{\mathrm{rig}}^*"] \end{tikzcd}$$

The top squares commute by flat pullback. The bottom squares commute by definition of the equivalence occurring in each vertical arrow. Let

$$\tilde{q} : \mathcal{S}^{\mathrm{rig}}_{g,r} \to [\mathcal{S}^{\mathrm{rig}}_{g,r}/\mu_{\mathbf{deg}}]$$

denote the quotient by the trivial action of $\mu_{\mathbf{deg}}$ on $\mathcal{S}^{\mathrm{rig}}_{g,r}$. The proof follows by observing that both of the above invariants are equal to

$$\mathrm{proj}_*\circ\mathrm{rig}_*\left(\mathrm{td}(-\mathbb{R}\pi_*\mathcal{V})\bar{\phi}_{\mathrm{HKR}}\circ\tilde{q}^*\circ\Phi_{\mathbb{L}\widetilde{\mathrm{rig}}^*\{-\alpha_0,\beta_0\}_*}(s_1,\dots,s_r)\right)$$

after scaling by either $(m_1\cdots m_r)/\deg(\mathrm{rig})$ in the case of $\Lambda_{g,r,0}$ or $\sigma_g(\underline{\boldsymbol{g}})/\deg(\mathrm{rig}\circ\mathrm{proj})$ in the case of $\lambda^{PV}_{g,r}$. For $\Lambda_{g,r,0}$ this follows from flat pullback of factorizations together with the definition of $\bar{\phi}_{\mathrm{HKR}}$ and Proposition 6.2.2. For $\lambda^{PV}_{g,r}$ one also uses Lemma 6.2.1 and the fact that $\mathbb{L}\tilde{t}^*K^{PV}$ is equal to $\mathbb{L}\widetilde{\mathrm{rig}}^*\{-\alpha_0,\beta_0\}$. □

6.3. **Comparison with other affine phases.**

Definition 6.3.1. *Define the GLSM $(V, G, \mathbb{C}^*_R, \theta, w)$ to be* equivalent *to $(V', G', \mathbb{C}^*_R, \theta', w')$ if the associated $\mathbb{C}^*_R$-equivariant LG spaces $([V /\!\!/_\theta G], \bar{w})$ and $([V' /\!\!/_{\theta'} G'], \bar{w}')$ are isomorphic where $\bar{w}$, $\bar{w}'$ are the induced regular functions on $[V /\!\!/_\theta G]$, $[V' /\!\!/_{\theta'} G']$, respectively.*

Recall in §6.2, we had a finite group G and a one-dimensional (not necessarily connected) commutative algebraic group Γ. Let us rename G by G', V by $V' = \mathbb{A}^m$, etc. This gives a GLSM $(\mathbb{A}^m, G', \mathbb{C}^*_R, 0, w')$. Let $(V, G, \mathbb{C}^*_R, \theta, w)$ be an equivalent abelian affine phase i.e. assume that $[V_1 /\!\!/_\theta G] \cong BG'$ and that the LG space obtained from $(V, G, \mathbb{C}^*_R, \theta, w)$ agrees with that of $(\mathbb{A}^m, G', \mathbb{C}^*_R, 0, w')$ in the sense of Definition 6.3.1. In this section, we will show that the GLSM invariants associated to $(V, G, \mathbb{C}^*_R, \theta, w)$ agree with those of $(\mathbb{A}^m, G', \mathbb{C}^*_R, 0, w')$ and consequently, with the FJRW invariants constructed in [PV16].

The moduli spaces $\mathcal{S}_{g,r} = LG_{g,r}(BG', d)$ and $LG_{g,r}([V_1 /\!\!/_\theta G], d)$ are isomorphic, as can be seen, for instance, from the construction in § 4. Thus we can restrict our attention to degree zero, where the moduli space is non-empty. Let $[A_1 \xrightarrow{d_1} B_1]$ be the resolution of $\mathbb{R}\pi_*(\mathcal{V}_1)$ and U the open Deligne–Mumford substack of $\mathrm{tot}(A_1)$ defined in Theorem 4.3.4, see also Diagram 4.16. Let $[\bar{A} \xrightarrow{\bar{d}} \bar{B}]$ denote the resolution of $\mathbb{R}\pi_*(\mathcal{V})$ over U (5.1). We may assume without loss of generality that this complex splits as $[\bar{A}_1 \oplus \bar{A}_2 \xrightarrow{\bar{d}_1, \bar{d}_2} \bar{B}_1 \oplus \bar{B}_2]$ where $[\bar{A}_i \xrightarrow{\bar{d}_i} \bar{B}_i]$ is a resolution of $\mathbb{R}\pi_*(\mathcal{V}_i)$. Recall that $\square$ lies in $\mathrm{tot}(\bar{A})$ over U, and $\{-\alpha, \beta\}$ is the Koszul factorization associated to the vector bundle E equal to the pullback of $\bar{B}$ with section β induced by $(\bar{d}_1, \bar{d}_2)$ and cosection α.

Let Z denote the restriction of $\bar{A}_2$ to $LG_{g,r}(BG', 0)$. Recall that $LG_{g,r}(BG', 0)$ is a closed substack of U. Note that $\bar{A}_1|_{LG_{g,r}(BG',0)}$ has a tautological section induced by the map $\pi_*(\mathcal{V}_1) \to \bar{A}_1$. This gives a closed immersion of $LG_{g,r}(BG', 0)$ into the total space $\mathrm{tot}(\bar{A}_1)$ over U. Combining this with the identity map on $\bar{A}_2$ yields a closed immersion

$$j : Z \hookrightarrow \square \subseteq \mathrm{tot}(\bar{A}).$$

Let E' denote the pullback of the vector bundle $\bar{B}_2$ to Z. The bundle E' has a natural section β' induced by $\bar{d}_2$, and a cosection α' defined by restricting α.

Proposition 6.3.2. *The factorizations $K' = \{-\alpha', \beta'\}$ and $K = \{-\alpha, \beta\}$ are related by pushforward.*

Proof. This follows from a slight variation of the arguments of §5.2. Consider the inclusion of vector bundles $\bar{B}_2 \subseteq E$ over $\square$. The quotient $E/\bar{B}_2$ is equal to $\bar{B}_1$, and

$$\beta \mod (\bar{B}_2) = \bar{\beta}_1.$$

The zero locus of $\bar{\beta}_1$ in $\mathrm{tot}(\bar{A})$ is the total space $\mathrm{tot}(\pi_*(\mathcal{V}_1) \oplus \bar{B}_2)$ over U. After restricting to $\square$, the zero locus of $\bar{\beta}_1$ is exactly the inclusion of Z by j. Note that on each connected component,

$$\begin{aligned} \dim Z &= \dim LG_{g,r}(BG', 0) + \mathrm{rank}\, \bar{A}_2 \\ &= \dim \mathfrak{M}_{g,r} + \mathrm{rank}\, \bar{A}_2 \\ &= \dim \mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma, 0)^{\circ}_{\omega^{\log}_{\mathfrak{C}}} + \chi(\mathbb{R}\pi_*(\mathcal{V}_1)) + \mathrm{rank}\, \bar{A}_2 \\ &= \dim \square - \mathrm{rank}\, E/\bar{B}_2. \end{aligned}$$

The first equality is from the definition of Z and the second equality is from [PV16, Proposition 3.2.6] or [FJR13, Theorem 2.2.6]. The third equality is due to the fact that $\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,0)^{\circ}_{\omega^{\log}_{\mathfrak{C}}}$ is a finite cover of $\mathfrak{M}^{\mathrm{orb}}_{g,r}(BG,0)^{\circ}$. Therefore $\dim \mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,0)^{\circ}_{\omega^{\log}_{\mathfrak{C}}} = \dim \mathfrak{M}^{\mathrm{orb}}_{g,r}(BG,0)^{\circ} = \dim \mathfrak{M}_{g,r} - \chi(\mathbb{R}\pi_*\mathcal{P}\times_G\mathfrak{g}) = \dim \mathfrak{M}_{g,r} - \chi(\mathbb{R}\pi_*\mathcal{V}_1)$, by orbifold Riemann–Roch [AGV08, Theorem 7.2.1]. The fourth equality is from the definitions of $\square$ and E.

We conclude that $\beta \mod (\bar{B}_2)$ is a regular section of $E/\bar{B}_2$. We observe further that the section $\beta|_Z \in \Gamma(Z, E')$ is equal to β', and that $\alpha' = \alpha \mod ((E')^{\perp}|_Z)$ by construction. It follows by Proposition 5.2.3 that

$$\mathbb{R}j_*(\{-\alpha', \beta'\}) = \{-\alpha, \beta\}. \tag{6.8}$$

$\square$

Theorem 6.3.3. *The invariants $\Lambda_{g,r,0}$ for the equivalent GLSMs $(V, G, \mathbb{C}^*_R, \theta, w)$ and $(\mathbb{A}^m, G', \mathbb{C}^*_R, \bar{\theta}' = 0, w')$ are equal.*

Proof. The proof follows a similar argument to Theorem 5.5.4. Observe that by construction, the factorization $\{-\alpha_0, \beta_0\}$ of Proposition 6.2.2 can be assumed without loss of generality to be exactly $\{-\alpha', \beta'\}$ from the previous proposition. Here $\mathrm{tot}(A_0)$ is identified with Z. Recall $\widetilde{U}$ denotes a desingularization of the closure of U, and let

$$\tilde{i} : \mathcal{S}_{g,r} \to \widetilde{U}$$

denote the inclusion. By the previous proposition,

$$\tilde{i}_* \circ (\Phi_{K'})_* = (\Phi_K)_*.$$

The GLSM invariants for $(\mathbb{A}^m, G', \mathbb{C}^*_R, 0, w')$ are by definition equal to

$$\mathrm{proj}_* \circ \tilde{i}_* \left(\mathrm{td}(\ominus \mathbb{R}\pi_*\mathcal{V}_2) \cup \left(\bar{\phi}_{\mathrm{HKR}} \circ (\Phi_{K'})_*(-)\right)\right)$$

up to a scaling, where proj_* is the projection to $\overline{\mathcal{M}}_{g,r}$. The invariants for $(V, G, \mathbb{C}^*_R, \theta, w)$ on the other hand are given by

$$\mathrm{proj}_* \left(\mathrm{td}(T_{\widetilde{U}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}} \ominus \mathbb{R}\pi_*\mathcal{V}) \cup \left(\bar{\phi}_{\mathrm{HKR}} \circ (\Phi_K)_*(-)\right)\right)$$

after the same scaling.

The relative tangent bundle of $\tilde{i}$ is equal to

$$\begin{aligned} &T_{\mathcal{S}_{g,r}/Z} + T_{Z/\square} + T_{\square/\tilde{U}} \\ \cong &(-\bar{A}_2) + (-\bar{B}_1) + (\bar{A}_1 + \bar{A}_2 - A_1) \\ \cong &\mathbb{R}\pi_*\mathcal{V}_1 - A_1, \end{aligned}$$

where the isomorphism $T_{Z/\square} \cong -\bar{B}_1$ follows from the fact proven in the previous proposition that $\bar{\beta}_1$ is a regular section of $\square$. Note further that the pullback of $T_{\widetilde{U}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}}$ via $\tilde{i}$ is equal to A_1. In particular

$$-\mathbb{R}\pi_*\mathcal{V}_2 = -\mathbb{R}\pi_*(\mathcal{V}) + \mathbb{R}\pi_*\mathcal{V}_1 = -\mathbb{R}\pi_*(\mathcal{V}) + \tilde{i}^*(T_{\widetilde{U}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}}) + T_{\mathcal{S}_{g,r}/\widetilde{U}}.$$

Applying the projection formula and the argument of Theorem 2.5.1 we conclude that

$$\begin{aligned}
&\tilde{i}_*\left(\operatorname{td}\left(\ominus\mathbb{R}\pi_*\mathcal{V}_2\right)\bar{\phi}_{\mathrm{HKR}}(-)\right) && (6.9)\\
=&\operatorname{td}\left(T_{\widetilde{U}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}}\ominus\mathbb{R}\pi_*\mathcal{V}\right)\tilde{i}_*\left(\operatorname{td}(T_{\mathcal{S}_{g,r}/\widetilde{U}})\bar{\phi}_{\mathrm{HKR}}(-)\right)\\
=&\operatorname{td}\left(T_{\widetilde{U}/\mathfrak{M}^{\mathrm{orb}}_{g,r}(B\Gamma,d)_{\omega^{\log}_{\mathfrak{C}}}}\ominus\mathbb{R}\pi_*\mathcal{V}\right)\bar{\phi}_{\mathrm{HKR}}\tilde{i}_*(-).
\end{aligned}$$

The result follows. □

Corollary 6.3.4. *Given an affine phase GLSM $(V, G, \mathbb{C}^*_R, \theta, w)$ constructed as above, let*

$$\bar{w} : [V/\!\!/_\theta G] = [\mathbb{A}^m/G'] \to \mathbb{A}^1$$

be the corresponding singularity. Then the GLSM invariant $\Lambda_{g,r,0}(s_1,\ldots,s_r)$ is equal to the reduced invariant of [PV16] *of the singularity after scaling by the factor* $(m_1\cdots m_r)\frac{\deg(\mathrm{proj})}{\sigma_g(\boldsymbol{g})}$.

Proof. This is an immediate consequence of the previous theorem and Theorem 6.2.3. □

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