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The abelian/nonabelian correspondence and Frobenius manifolds

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Abstract. We propose an approach via Frobenius manifolds to the study (began in [BCK2]) of the relation between rational Gromov–Witten invariants of nonabelian quotients $X/\!/\mathbf{G}$ and those of the corresponding "abelianized" quotients $X/\!/\mathbf{T}$, for $\mathbf{T}$ a maximal torus in $\mathbf{G}$. The ensuing conjecture expresses the Gromov–Witten potential of $X/\!/\mathbf{G}$ in terms of the potential of $X/\!/\mathbf{T}$. We prove this conjecture when the nonabelian quotients are partial flag manifolds.

1. Introduction

1.1. The paper [BCK2] conjectures a correspondence between the genus zero Gromov–Witten invariants of nonsingular projective GIT quotients $X/\!/\mathbf{G}$ and $X/\!/\mathbf{T}$, for $\mathbf{G}$ a complex reductive Lie group with a linearized action on a projective manifold X and $\mathbf{T}$ a maximal torus in $\mathbf{G}$. The correspondence expresses (descendant) Gromov–Witten invariants of $X/\!/\mathbf{G}$ in terms of Gromov–Witten invariants of $X/\!/\mathbf{T}$ *twisted* by (the top Chern class of) a certain decomposable vector bundle on $X/\!/\mathbf{T}$.

Our main goal in this paper is to give a natural reformulation of the correspondence in terms of the Frobenius structures describing the (big) quantum cohomology rings $QH^*(X/\!/\mathbf{G}, \mathbb{C})$ and $QH^*(X/\!/\mathbf{T}, \mathbb{C})$. This is accomplished in Sect. 3. To explain it, recall that a given cohomology class $\sigma \in H^*(X/\!/\mathbf{G})$ can be lifted to a class $\tilde{\sigma}$ (of the same degree) in the Weyl group invariant subspace $H^*(X/\!/\mathbf{T})^{\mathbf{W}}$. Such a lift is not unique, however, if ω is the fundamental $\mathbf{W}$-anti-invariant class, then $\tilde{\sigma} \cup \omega$ is uniquely determined by σ. Moreover, by results of Ellingsrud and Strømme when $X = \mathbb{P}^N$, and later Martin in full generality, this identification respects cup products:

$$(\widetilde{\sigma \cup_{X//\mathbf{G}} \sigma'}) \cup \omega = \tilde{\sigma} \cup (\tilde{\sigma}' \cup \omega) \in H^*(X//\mathbf{T}).$$

A naive guess might be that the identification respects quantum products as well, that is,

$$(\widetilde{\sigma \star_{X//\mathbf{G}} \sigma'}) \cup \omega = \tilde{\sigma} \star_{X//\mathbf{T}} (\tilde{\sigma}' \cup \omega),$$

after an appropriate specialization of quantum parameters. Indeed, as shown in [BCK1], Theorem 2.5, this is the case for *small* quantum products when $X//\mathbf{G}$ is a Grassmannian. At the level of Gromov–Witten invariants, this would translate into an appealing identity of the form

(1.1.1)
$$\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \sigma_n \rangle_{0,n,\beta}^{X//\mathbf{G}} = \pm \sum_{\tilde{\beta} \mapsto \beta} \langle \tilde{\sigma}_1, \ldots, \tilde{\sigma}_{n-2}, \tilde{\sigma}_{n-1} \cup \omega, \tilde{\sigma}_n \cup \omega \rangle_{0,n,\tilde{\beta}}^{X//\mathbf{T}}.$$

It is not hard to convince oneself, however, that this fails for big quantum cohomology (already for the Grassmannian $Grass(2, 4)$), and that it has no reason to be true in general even for small quantum cohomology. Instead, we conjecture a generalization to quantum cohomology as follows:

Fix a lifting $\tilde{\bullet}$ of $H^*(X//\mathbf{G})$ to a subspace $U \subset H^*(X//\mathbf{T})^{\mathbf{W}}$. Let $\{t_i\}$ be the coordinates on $H^*(X//\mathbf{G})$, corresponding to a choice of basis, and let $\{\tilde{t}_i\}$ be the coordinates on U corresponding to the lifted basis. Let $N(X//\mathbf{G})$ and $N(X//\mathbf{T})$ be the Novikov rings for the two quotients.

The quantum product in $QH^*(X//\mathbf{G}, \mathbb{C})$ is a $N(X//\mathbf{G})[[t]]$-linear product on $H^*(X//\mathbf{G}, \mathbb{C}) \otimes_{\mathbb{C}} N(X//\mathbf{G})[[t]]$, while the quantum product in $QH^*(X//\mathbf{T}, \mathbb{C})$ is a $N(X//\mathbf{T})[[\tilde{t}, y]]$-linear product on $H^*(X//\mathbf{T}, \mathbb{C}) \otimes_{\mathbb{C}} N(X//\mathbf{T})[[\tilde{t}, y]]$, where $(\tilde{t}, y)$ is an extension of $\tilde{t}$ to coordinates on the entire $H^*(X//\mathbf{T}, \mathbb{C})$.

There is a natural specialization of Novikov variables $p : N(X//\mathbf{T}) \to N(X//\mathbf{G})$ which takes into account that there are more curve classes on $X//\mathbf{T}$. We denote by "$\star$" the quantum product on $X//\mathbf{T}$ with the Novikov variables specialized via p. Given $\sigma, \sigma' \in H^*(X//\mathbf{G})$, there are classes $\xi, \xi' \in U \otimes_{\mathbb{C}} N(X//\mathbf{G})[[\ \tilde{t}\ ]]$, uniquely determined by $\xi \star \omega = \tilde{\sigma} \cup \omega$ and $\xi' \star \omega = \tilde{\sigma}' \cup \omega$ respectively.

Conjecture. There is an equality

$$((\widetilde{\sigma \star_{X//\mathbf{G}} \sigma'}) \cup \omega)(t) = \xi \star \xi' \star \omega\ (\tilde{t}, 0) = (\xi \star (\tilde{\sigma}' \cup \omega))(\tilde{t}, 0),$$

after an explicit change of variable $\tilde{t} = \tilde{t}(t)$.

At the level of Gromov–Witten invariants, the Conjecture says that the right-hand side of the naive formula (1.1.1) receives a correction term which is a sum of products of invariants of $X//\mathbf{T}$ of the same type (see the appendix for a discussion and some examples).

We should warn the reader that the above formulation is a translation of the actual Conjecture 3.7.1 in the body of the paper, which is stated in the conceptual framework of Frobenius structures. It is in this framework that one is naturally lead to the conjecture. Indeed, if N is the formal germ of the affine space over $N(X//\mathbf{G})$ associated to the subspace U, the general machinery of the infinitesimal period mapping in the theory of Frobenius–Saito structures (see e.g., [Sab]) gives a canonical Frobenius manifold structure on N. It is induced by the primitive homogeneous section ω of the (trivial) bundle with fiber the anti-invariant subspace $H^*(X//\mathbf{T})^a$ over N, together with the restriction to this bundle (in an appropriate sense) of the Frobenius structure on $H^*(X//\mathbf{T})$. Our conjecture says that this new Frobenius manifold is identified with the Frobenius manifold given by the Gromov–Witten theory of $X//\mathbf{G}$. The new flat metric ${}^{\omega}g$ on the sheaf Θ_N of vector fields satisfies

$$ {}^{\omega}g(\tilde{\sigma}, \tilde{\sigma}') = g(\tilde{\sigma} \star \omega, \tilde{\sigma}' \star \omega). $$

It follows that the coordinates $\{\tilde{t}_i\}$ on N provided by lifting are *not* flat for the new Frobenius structure, or, equivalently, the liftings $\tilde{\sigma}$ are not horizontal vector fields. The vector fields ξ, ξ' appearing in the statement of the conjecture are precisely the horizontal vector fields corresponding to σ, σ' under the identification of flat coordinates of Frobenius structures. This identification of coordinates is the change variable $\tilde{t} = \tilde{t}(t)$.

In fact, we treat a more general situation in Sect. 3, by considering the *equivariant* Gromov–Witten theories in the presence of compatible actions of an additional torus $\mathbf{S}$ on $X//\mathbf{G}$ and $X//\mathbf{T}$. The corresponding Frobenius structures are more general than the ones considered in [Sab], as they lack an Euler vector field. However, a suitable modification of the notion of Euler vector field allows the application of the theory of infinitesimal period mappings in this case as well. We give an exposition of the relevant facts in Sects. 2.2–2.3.

This generalization is needed in Sect. 4, where we prove, by using reconstruction theorems for Gromov–Witten invariants (extended to the equivariant setting), that the conjecture above can be reduced in many cases to the abelian/nonabelian correspondence for *small* J-functions from [BCK2]. In particular, the following result is obtained:

Theorem. *Let $Fl = Fl(k_1, \ldots, k_r, n)$ be the flag manifold parameterizing flags of subspaces $\{\mathbb{C}^{k_1} \subset \cdots \subset \mathbb{C}^{k_r} \subset \mathbb{C}^n\}$, viewed as a GIT quotient $\mathbb{P}^l//\mathbf{G}$ for appropriate $l, \mathbf{G}$. Denote by Y the toric variety which is the corresponding abelian quotient $\mathbb{P}^l//\mathbf{T}$ (cf.* [BCK2]*). Then the conjecture is true for the pair (Fl, Y).*

The theorem implies that the genus zero Gromov–Witten invariants of a flag manifold (with any number of insertions) can be expressed in terms of Gromov–Witten invariants of the associated toric variety Y. In an appendix we write down explicit formulae in the simplest case of the Grassmannian $Grass(k, n)$, for which the abelian quotient is the product of k copies of $\mathbb{P}^{n-1}$.

In Sect. 5, we obtain an equivalent formulation (5.3.4) of the conjecture in terms of (big) J-functions of $X//\mathbf{T}$ and $X//\mathbf{G}$. It generalizes Conjecture 4.3 of [BCK2] and, by the above theorem, it holds for type A flag manifolds.

Finally, in Sect. 6 we extend the abelian/nonabelian correspondence to include Gromov–Witten invariants with an additional twist by homogeneous vector bundles. As an application, we describe the J-function of a generalized flag manifold for a simple complex Lie group of type B, C, or D as the twisted J-function of the abelianization of the corresponding flag manifold of type A.

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2. Preliminaries on Frobenius structures

2.1. Formal Frobenius manifolds from Gromov–Witten theory. Let R be a $\mathbb{C}$-algebra and let K be a free R-module of rank m. We think of K as the affine m-space over R (precisely, the spectrum of the symmetric algebra of the dual module). Let $M := Spf(R[[K^\vee]])$ be the formal completion of K at the origin. M is a formal manifold over R. We denote by Θ_M its formal relative tangent sheaf over R. Note that it is canonically identified with $K \otimes_R \mathcal{O}_M$.

Definition 2.1.1. *The data $(M, \star, g, e, \mathfrak{E})$ is called a (conformal, even) formal Frobenius manifold over R if the following properties hold:*

- *g is an $\mathcal{O}_M$-linear, nondegenerate pairing such that its metric connection ∇ is flat*
- *$\star$ is an $\mathcal{O}_M$-linear, associative, commutative product on Θ_M*
- *e is a formal vector field on M over R which is the identity for the product $\star$, and such that $\nabla e = 0$*
- *∇c is symmetric, where the tensor c is defined by $c(u, v, w) = g(u \star v, w)$*
- *$\mathfrak{E}$ is a formal vector field on M over R satisfying*

$$\mathcal{L}_{\mathfrak{E}}(g) = Dg, \quad \mathcal{L}_{\mathfrak{E}}(\star) = \star, \quad \mathcal{L}_{\mathfrak{E}}(e) = -e,$$

where $\mathcal{L}_{\mathfrak{E}}$ denotes the Lie derivative and $D \in \mathbb{C}$ is a constant.

The fourth condition implies that there is a formal function F on M (the *potential* of the Frobenius manifold) such that the tensor c is given by the third derivatives of F in flat coordinates, and then associativity of $\star$ translates into the WDVV equations for F. The vector field $\mathfrak{E}$ is called an *Euler vector field*.

We recall here the formal Frobenius manifold structures determined by the genus zero GW-theories (ordinary and equivariant) of a projective manifold endowed with an action of an algebraic complex torus $\mathbf{S} \cong (\mathbb{C}^*)^\ell$. Detailed expositions can be found in [LP2] or [Man], to which we refer the reader.

Let Y be a smooth projective variety over $\mathbb{C}$. We assume for simplicity that $H_2(Y, \mathbb{Z})$ is torsion-free and that the odd cohomology $H^{2*+1}(Y, \mathbb{C})$ vanishes. We denote by $N(Y)$ the *Novikov ring* of Y. It can be described as the $\mathbb{C}$-algebra of "power series" $\{\sum_{\beta \in NE_1} c_\beta Q^\beta | c_\beta \in \mathbb{C}\}$, where $NE_1 \subset H_2(Y, \mathbb{Z})$ is the semigroup of effective curve classes.

The genus zero Gromov–Witten theory of Y determines a formal Frobenius manifold over $R = N(Y)$. We take

$$K = N(Y) \otimes_{\mathbb{C}} H^*(Y, \mathbb{C}),$$

so that $M = Spf(N(Y)[[K^\vee]])$. The metric g is given by the intersection pairing:

$$g(\gamma, \gamma') = \int_Y \gamma \cup \gamma'.$$

Let $\{1 = \gamma_0, \gamma_1, \ldots, \gamma_r, \gamma_{r+1}, \ldots, \gamma_{m-1}\}$ be a basis of $H^*(Y, \mathbb{C})$ consisting of integral homogeneous classes, such that $\gamma_1, \ldots, \gamma_r$ form a basis of H^2. We write $\sigma = \sum t_i \gamma_i$ for a general cohomology class on Y. The functions t_i give flat coordinates on M. A potential function is defined using the genus zero Gromov–Witten invariants of Y

$$F(Q, t) := \sum_{\beta \in NE_1} \sum_{n \geq 0} Q^\beta \frac{1}{n!} \langle \underbrace{\sigma, \ldots, \sigma}_{n} \rangle_{0,n,\beta},$$

where the unstable terms with $\beta = 0,\ n \leq 2$ are omitted in the sum. The tensor c is given in flat coordinates by

$$c_{ijk} = \partial_{t_i} \partial_{t_j} \partial_{t_k} F$$

and the product $\star$ is called the *big quantum product*. The unit vector field e is given by the class $\gamma_0 = 1$.

The following notation is customary:

$$\langle\langle \sigma_1, \ldots, \sigma_r \rangle\rangle = \sum_{\beta \in NE_1} \sum_{n} Q^\beta \frac{1}{n!} \langle \sigma_1, \ldots, \sigma_r, \underbrace{\sigma, \ldots, \sigma}_{n} \rangle_{0,n+r,\beta},$$

where $\sigma_j \in H^*(Y, \mathbb{C})$ are given cohomology classes and $\sigma = \sum t_i \gamma_i$ is the general element in $H^*(Y, \mathbb{C})$ (so that $\langle\langle\ \rangle\rangle = F$). We extend this double bracket $\mathcal{O}_M$-linearly to general vector fields $\sigma_1, \dots, \sigma_r$. It is easy to see that for any vector field ξ we have

$$\nabla_\xi(F) = \langle\langle \xi \rangle\rangle.$$

In particular, since $\nabla_{\partial_{t_i}} \partial_{t_j} = 0$, the quantum product can be written in our chosen basis

$$\gamma_i \star \gamma_j = \sum_k \langle\langle \gamma_i, \gamma_j, \gamma_k \rangle\rangle \gamma_k^\vee$$

where $\gamma_k^\vee = \sum_l g^{kl} \gamma_l$ with (g^{kl}) the inverse matrix of the metric g.

The divisor axiom for Gromov–Witten invariants implies that the Gromov–Witten potential has the special form

$$F = F_{cl} + \sum_{\beta \in NE_1, \beta \neq 0} Q^\beta e^{\beta \cdot t_{\rm small}} F_\beta, \tag{2.1.1}$$

with F_{cl} a cubic polynomial in the t_i's and $F_\beta \in \mathbb{C}[[t_{r+1}, \dots, t_{m-1}]]$ formal power series in the *non-divisorial* coordinates. Here we use the notation $\beta \cdot t_{\rm small}$ for the intersection index of β with the general H^2-class,

$$\beta \cdot t_{\rm small} := \int_\beta \sum_{i=1}^r t_i \gamma_i.$$

We will also use the notation F_q for $F - F_{cl}$.

Assume now that Y is acted upon by the torus $\mathbf{S} \cong (\mathbb{C}^*)^\ell$. The equivariant cohomology $H^*_{\mathbf{S}}(Y, \mathbb{C})$ is a module over the polynomial ring

$$H^*_{\mathbf{S}}(pt) = H^*(B\mathbf{S}) \cong \mathbb{C}[\lambda_1, \dots, \lambda_\ell],$$

and it is in fact a free module by [Gin]. Taking

$$R = N(Y)[\lambda] := N(Y) \otimes_{\mathbb{C}} \mathbb{C}[\lambda_1, \dots, \lambda_\ell]$$

and

$$K_{\mathbf{S}} = R \otimes_{\mathbb{C}[\lambda_1, \dots, \lambda_\ell]} H^*_{\mathbf{S}}(Y, \mathbb{C})$$

we get similarly a formal Frobenius manifold over R. The metric g is now given by the ($\mathbb{C}[\lambda_1, \dots, \lambda_\ell]$-valued) equivariant intersection pairing, while in F the GW-invariants are replaced by their $\mathbf{S}$-equivariant counterparts. The unit vector field and equivariant big quantum product are obtained analogously.

Localization with respect to $\mathbf{S}$ determines yet another Frobenius structure. Consider the localization of $H^*(B\mathbf{S})$, i.e., the field of fractions $\mathbb{C}(\lambda_1, \dots, \lambda_\ell)$, and set

$$N(Y)[\lambda]_{(\lambda)} = N(Y) \otimes_{\mathbb{C}} \mathbb{C}(\lambda_1, \dots, \lambda_\ell)$$
$$K^*_{\mathbf{S}} = N(Y)[\lambda]_{(\lambda)} \otimes_{N(Y)[\lambda]} K_{\mathbf{S}}.$$

Taking $M = Spf(N(Y)[\lambda]_{(\lambda)}[[K_{\mathbf{S}}^{*\vee}]])$ with the localized equivariant metric, potential function, and unit vector field determines a formal Frobenius manifold over $N(Y)[\lambda]_{(\lambda)}$ (in other words, we simply consider the Frobenius structure induced by base change via $N(Y) \to N(Y)[\lambda]_{(\lambda)}$).

In both the equivariant and localized equivariant cases the potential function in flat coordinates t has the special form (2.1.1), with $F_\beta \in \mathbb{C}[\lambda][[t_{r+1}, \ldots, t_{m-1}]]$.

Finally, we discuss the Euler vector fields. The Frobenius manifold defined by the (nonequivariant) Gromov–Witten theory of Y is conformal: the Euler vector field (with $D = 2 - \dim(Y)$) is explicitly

$$\mathfrak{E} = \sum_{i=0}^{m-1} \left(1 - \frac{\mathrm{cdeg}\gamma_i}{2}\right) t_i \partial_{t_i} + c_1(TY).$$

Here "cdeg" is the cohomological degree.

Consider the $\mathbf{S}$-equivariant version of this vector field

$$\mathfrak{E} = \sum_{i=0}^{m-1} \left(1 - \frac{\mathrm{cdeg}\gamma_i}{2}\right) t_i \partial_{t_i} + c_1^{\mathbf{S}}(TY)$$

with γ_i's now forming a basis of $H_{\mathbf{S}}^*(Y, \mathbb{C})$ over $H^*(B\mathbf{S})$. $\mathfrak{E}$ does not give a conformal structure (because equivariant Gromov–Witten invariants do not satisfy a dimension constraint). Nevertheless, we consider a variant of the Euler vector field in this context as well, by relaxing the requirement of linearity over $N(Y) \otimes \mathbb{C}[\lambda]$ and will define below an Euler vector field $\mathfrak{E}_{\mathbf{S}}$ as an $N(Y)$-derivation on $\mathcal{O}_M$ (that is, an $N(Y)$-derivation of $K_{\mathbf{S}}^\vee$ into itself). The flat coordinates $\{t_i\}$, together with $\{\lambda_1, \ldots, \lambda_\ell\}$ form a coordinate system on M over $Spec(N(Y))$. Therefore

$$\mathcal{E}_{\mathbf{S}} := \sum_{i=1}^{\ell} \lambda_i \partial_{\lambda_i}$$

is a well-defined "absolute" vector field (i.e., $N(Y)$-linear derivation) and acts by Lie bracket on the relative vector fields Θ_M. Put

$$\mathfrak{E}_{\mathbf{S}} := \mathfrak{E} + \mathcal{E}_{\mathbf{S}}.$$

If $\eta \in \Theta_M$ is any relative vector field, then the commutator $[\mathfrak{E}_{\mathbf{S}}, \eta]$ is also in Θ_M. Hence Lie derivatives of tensors on Θ_M are well defined. The vector field $\mathfrak{E}_{\mathbf{S}}$ will still satisfy the conditions in Definition 2.1.1, again with $D = 2 - \dim(Y)$. The same $\mathfrak{E}_{\mathbf{S}}$ will be used for the localized structure as well.

We have

$$\mathcal{L}_{\mathfrak{E}_{\mathbf{S}}}(\partial_{t_i}) = \left(-1 + \frac{\mathrm{cdeg}\gamma_i}{2}\right)\partial_{t_i}, \ \mathcal{L}_{\mathfrak{E}_{\mathbf{S}}}(\lambda_i) = \lambda_i, \ \mathcal{L}_{\mathfrak{E}_{\mathbf{S}}}(F) = (3 - \dim(Y))F.$$

2.2. S-equivariant Frobenius manifolds over R. We extend the construction of Frobenius manifold through an infinitesimal period mapping to the previous setting. Let M be as above, with $\mathcal{O}_M = R[[K_{\mathbf{S}}^{*\vee}]]$ and $R = N(Y)[\lambda]$ or $R = N(Y)[\lambda]_{(\lambda)}$. Let E be a free $\mathcal{O}_M$-module of finite rank. An $\mathbf{S}$-*conformal connection* on E consists of a pair $\widetilde{\nabla} = (\nabla, \widetilde{\nabla}_{\mathcal{E}_{\mathbf{S}}})$, where ∇ is an R-connection on E and $\widetilde{\nabla}_{\mathcal{E}_{\mathbf{S}}}$ is a $N(Y)$-linear derivation satisfying $\widetilde{\nabla}_{\mathcal{E}_{\mathbf{S}}}(\varphi e) = \mathcal{L}_{\mathcal{E}_{\mathbf{S}}}(\varphi)e + \varphi\widetilde{\nabla}_{\mathcal{E}_{\mathbf{S}}}e$ for any $e \in E$ and $\varphi \in \mathcal{O}_M$. We say that $\widetilde{\nabla}$ is *flat* if ∇ is flat and for any vector field $\xi \in \Theta_M$, $[\widetilde{\nabla}_{\mathcal{E}_{\mathbf{S}}}, \nabla_\xi] = \nabla_{[\mathcal{E}_{\mathbf{S}},\xi]}$. In coordinates (t_i) defined from an $N(Y)$-basis of K, the previous condition is equivalent to the pairwise commutation of the operators $\nabla_{\partial_{t_i}}$ and $\widetilde{\nabla}_{\mathcal{E}_{\mathbf{S}}}$. Such a connection $\widetilde{\nabla}$ extends in a natural way to a similar object on $\hom_{\mathcal{O}_M}(E, E)$.

An $\mathbf{S}$-equivariant *pre-Saito structure* $(M, E, \widetilde{\nabla}, \Phi, R_0, g)$ of weight w over M consists of

- a free $\mathcal{O}_M$-module E of finite rank with a flat $\mathbf{S}$-conformal connection $\widetilde{\nabla}$,
- $\mathcal{O}_M$-linear morphisms $\Phi : \Theta_M \otimes_{\mathcal{O}_M} E \to E$ and $R_0 : E \to E$,
- an $\mathcal{O}_M$-bilinear form g on E,

satisfying, when expressed in coordinates (t_i), the following relations for all i, j:

$$\nabla_{\partial_{t_i}}(\Phi_{\partial_{t_j}}) = \nabla_{\partial_{t_j}}(\Phi_{\partial_{t_i}}), \quad [\Phi_{\partial_{t_i}}, \Phi_{\partial_{t_j}}] = 0, \quad [R_0, \Phi_{\partial_{t_i}}] = 0,$$
$$\Phi_{\partial_{t_i}} - \widetilde{\nabla}_{\mathcal{E}_{\mathbf{S}}}(\Phi_{\partial_{t_i}}) + \nabla_{\partial_{t_i}}(R_0) = 0,$$
$$\nabla(g) = 0, \quad \widetilde{\nabla}_{\mathcal{E}_{\mathbf{S}}}(g) = -wg, \quad \Phi^*_{\partial_{t_i}} = \Phi_{\partial_{t_i}}, \quad R_0^* = R_0,$$

where * means taking the g-adjoint and $\widetilde{\nabla}_{\mathcal{E}_{\mathbf{S}}}(g)$ is defined as usual by the formula $\widetilde{\nabla}_{\mathcal{E}_{\mathbf{S}}}(g)(\xi, \eta) = \mathcal{L}_{\mathcal{E}_{\mathbf{S}}}(g(\xi, \eta)) - g(\widetilde{\nabla}_{\mathcal{E}_{\mathbf{S}}}\xi, \eta) - g(\xi, \widetilde{\nabla}_{\mathcal{E}_{\mathbf{S}}}\eta)$.

The pull-back of an $\mathbf{S}$-equivariant pre-Saito structure by a morphism $f : N \to M$ is well-defined only for morphisms f^* which commute with $\mathcal{L}_{\mathcal{E}_{\mathbf{S}}}$.

The definition of an $\mathbf{S}$-*equivariant Frobenius manifold over* R is a variant of Definition 2.1.1: With the same data $(M, \star, g, e, \mathfrak{E})$, we set $\mathfrak{E}_{\mathbf{S}} = \mathfrak{E} + \mathcal{E}_{\mathbf{S}}$, and we replace the homogeneity conditions by the following ones:

$$\mathcal{L}_{\mathfrak{E}_{\mathbf{S}}}(g) = Dg, \quad \mathcal{L}_{\mathfrak{E}_{\mathbf{S}}}(\star) = \star, \quad \mathcal{L}_{\mathfrak{E}_{\mathbf{S}}}(e) = -e.$$

Let $(M, E, \widetilde{\nabla}, \Phi, R_0, g)$ be an $\mathbf{S}$-equivariant *pre-Saito structure* of weight w and let ω be a ∇-horizontal section of E. It defines an $\mathcal{O}_M$-linear morphism $\varphi_\omega : \Theta_M \to E$ by $\xi \mapsto -\Phi_\xi(\omega)$. We say that such a section ω of E is

(1) *primitive* if the associated period mapping $\varphi_\omega : \Theta_M \to E$ is an isomorphism,
(2) *homogeneous* of degree $q \in \mathbb{C}$ if $\widetilde{\nabla}_{\mathcal{E}_{\mathbf{S}}}\omega = q\omega$.

The data of an $\mathbf{S}$-equivariant pre-Saito structure and of a homogeneous primitive section ω is called an $\mathbf{S}$-equivariant Saito structure. As in [Sab, §4.3] and following K. Saito, we obtain the following results.

If ω is primitive and homogeneous, φ_ω induces a flat, torsion-free, R-connection ${}^\omega\nabla := \varphi_\omega^{-1}\nabla\varphi_\omega$ on Θ_M, and an associative and commutative $\mathcal{O}_M$-bilinear product $\star$ by $\xi \star \eta = -\Phi_\xi(\varphi_\omega(\eta))$, with $e = \varphi_\omega^{-1}(\omega)$ as unit, and ${}^\omega\nabla e = 0$. Moreover, ${}^\omega\nabla$ is the metric connection attached to the metric ${}^\omega g$ on Θ_M obtained from g through φ_ω, and ${}^\omega\nabla$ is $\mathbf{S}$-conformal and flat as such, setting $\widetilde{{}^\omega\nabla}_{\mathcal{E}_\mathbf{S}} = \varphi_\omega^{-1} \circ \widetilde{\nabla}_{\mathcal{E}_\mathbf{S}} \circ \varphi_\omega - \mathrm{Id}$.

The Euler field is $\mathfrak{E} = \varphi_\omega^{-1}(R_0(\omega))$. It is therefore a section of Θ_M. We have ${}^\omega\nabla\mathfrak{E} = \mathcal{L}_{\mathcal{E}_\mathbf{S}} - \widetilde{{}^\omega\nabla}_{\mathcal{E}_\mathbf{S}} + q\mathrm{Id}$. In particular, ${}^\omega\nabla({}^\omega\nabla\mathfrak{E}) = 0$.

If we put $D = 2q + 2 - w$, and if we set as above $\mathfrak{E}_\mathbf{S} = \mathfrak{E} + \mathcal{E}_\mathbf{S}$, we get

$$\mathcal{L}_{\mathfrak{E}_\mathbf{S}}(e) = -e, \quad \mathcal{L}_{\mathfrak{E}_\mathbf{S}}(\star) = \star, \quad \mathcal{L}_{\mathfrak{E}_\mathbf{S}}({}^\omega g) = D \cdot {}^\omega g.$$

Given an $\mathbf{S}$-equivariant pre-Saito structure $(M, E, \widetilde{\nabla}, \Phi, R_0, g)$ of weight w, the datum of a homogeneous primitive section ω of E having weight q induces on M, through φ_ω, the structure of a $\mathbf{S}$-equivariant Frobenius manifold of weight $D = 2q + 2 - w$.

Conversely, any $\mathbf{S}$-equivariant Frobenius manifold $(M, \star, g, e, \mathfrak{E})$ defines an $\mathbf{S}$-equivariant pre-Saito structure $(M, \Theta_M, \widetilde{\nabla}, \Phi, R_0, g)$ having e as homogeneous primitive form.

For instance, to give the correspondence $(M, \star, g, e, \mathfrak{E}) \mapsto (M, \Theta_M, \widetilde{\nabla}, \Phi, R_0, g)$ we take ∇ to be the Levi–Civita connection of g, and

$$\widetilde{\nabla}_{\mathcal{E}_\mathbf{S}} = \mathrm{Id} + \mathcal{L}_{\mathcal{E}_\mathbf{S}} - \nabla\mathfrak{E}, \quad \Phi_\xi(\eta) = -(\xi \star \eta),$$
$$R_0 = \mathfrak{E}\,\star = -\Phi_\mathfrak{E}, \quad q = 0, \quad w = 2 - D.$$

2.3. S-Equivariant Frobenius manifolds with finite group action. Let us consider an $\mathbf{S}$-equivariant Frobenius manifold $(M, \star, g, e, \mathfrak{E})$ of weight D over R. Let W be a finite group which acts by relative automorphisms on M, hence on Θ_M, in a compatible way with the $\mathbf{S}$-equivariant Frobenius structure. (To be precise, we assume that W acts trivially on R, and on M by automorphisms preserving the map to $Spec(R)$.) In particular, the action of W on Θ_M commutes with $\mathcal{L}_{\mathcal{E}_\mathbf{S}}$.

Let M^W be the fixed set of W on M. Then W acts by $\mathcal{O}_{M^W}$-linear isomorphisms on $\Theta_M|_{M^W}$. Moreover, the fixed set M^W is a smooth subscheme of M over R and the fixed bundle $(\Theta_M|_{M^W})^W$ is equal to Θ_{M^W}.

Let us moreover assume that W is equipped with a *non trivial* character $\mathrm{sgn} : W \to \{\pm 1\}$. We denote by $a : \Theta_M|_{M^W} \to \Theta_M|_{M^W}$ the antisymmetrization morphism and by E its image. Then E is a locally free $\mathcal{O}_{M^W}$-submodule of $\Theta_M|_{M^W}$ and we have a decomposition $\Theta_M|_{M^W} = E \oplus \ker a$. This decomposition is g-orthogonal, as $g(a\xi, a\eta) = g(\xi, \eta)$ for any ξ, η and g restricted to E is nondegenerate.

As the inclusion $M^W \hookrightarrow M$ commutes with $\mathcal{L}_{\mathcal{E}_\mathbf{S}}$, one can restrict to M^W the $\mathbf{S}$-equivariant pre-Saito structure associated to $(M, \star, g, e, \mathfrak{E})$ to get such an object with corresponding bundle $\Theta_M|_{M^W}$. One can moreover induce this structure on the $\mathcal{O}_{M^W}$-module E, as the following operators leave E invariant:

- the connection ∇ (i.e., $\nabla_\xi\eta$ is a section of E whenever ξ is a section of Θ_{M^W} and η a section of E),
- the Higgs field Φ, (i.e., $\xi \star \eta = -\Phi_\xi\eta$ is a section of E whenever ξ is a section of Θ_{M^W} and η a section of E),
- the operator $R_0 = -\Phi_{\mathfrak{E}} = \mathfrak{E}\,\star$,
- the operator $\widetilde{\nabla}_{\mathscr{E}_{\mathbf{S}}} = \mathrm{Id} + \mathscr{L}_{\mathscr{E}_{\mathbf{S}}} - \nabla\mathfrak{E}$ (i.e., $\nabla_\eta\mathfrak{E}$ is a section of E whenever η is a section of E).

The following is then clear:

Lemma 2.3.1. *The tuple $(M^W, E, \widetilde{\nabla}, \Phi, R_0, g)$ defines an $\mathbf{S}$-equivariant pre-Saito structure of weight $w = 2 - D$ on M^W.*

Proposition 2.3.2. *Let us assume that there exists a section ω of $E \subset \Theta_M|_{M^W}$ which is ∇-horizontal and an eigenvector of $\widetilde{\nabla}_{\mathscr{E}_{\mathbf{S}}}$ (acting on E or on $\Theta_M|_{M^W}$) and such that the morphism*

$$\begin{aligned} \Theta_{M^W} &\longrightarrow E \\ \xi &\longmapsto \xi \star \omega \end{aligned}$$

is onto. Then, any smooth formal subscheme $N \subset M^W$ over R defined by an ideal invariant under $\mathscr{L}_{\mathscr{E}_{\mathbf{S}}}$ and such that the induced morphism $\Theta_N \to E|_N$ is an isomorphism comes equipped with a natural structure of an $\mathbf{S}$-equivariant Frobenius manifold of weight D.

Proof. We restrict the $\mathbf{S}$-equivariant pre-Saito structure $(M^W, E, \widetilde{\nabla}, \Phi, R_0, g)$ to N to get an object $(N, E|_N, \widetilde{\nabla}, \Phi, R_0, g)$ of the same kind. Then, as $\omega|_N$ is a ∇-horizontal section of $E|_N$ and as the morphism $\Theta_N \to E|_N$ given by $\xi \mapsto \xi\star\omega|_N = \varphi_\omega(\xi)$ is an isomorphism, ω is primitive. Moreover, ω is homogeneous in E hence $\omega|_N$ is so in $E|_N$. One can then apply the correspondence of Sect. 2.2. ☐

Some properties of the $\mathbf{S}$-equivariant Frobenius manifold structure on N. Abusing notation, we denote by $-\star\omega^{-1}$ the inverse map of the isomorphism $\star\,\omega : \Theta_N \to E|_N$, i.e., we will write it also as operating on the right.

We denote by ${}^\omega g$, ${}^\omega\nabla$ the metric and connection on Θ_N coming from that on $E|_N$, and by $\circ$ the product on Θ_N induced by the Higgs field on $E|_N$.

(1) Let ξ, η be sections of Θ_N. The product $\xi\star\eta$ in $\Theta_M|_N$ may not be a section of Θ_N (it is only a section of $\Theta_{M^W}|_N$). We have $[\xi \star \eta - \xi \circ \eta] \star \omega = 0$. In fact, the composition

$$\Theta_{M^W}\Big|_N \xrightarrow{\star\omega} E|_N \xrightarrow{\star\omega^{-1}} \Theta_N$$

induces a projection $\Theta_{M^W}|_N \to \Theta_N$, and $\xi \circ \eta$ is nothing else but the projection of $\xi \star \eta$ on Θ_N, so that we have the formula

$$\xi \circ \eta = (\xi \star \eta \star \omega) \star \omega^{-1}.$$

(2) Let us assume that we can find N such that the unit field e is *tangent* to N. This condition does not lead to a contradiction, as $e \star \omega = \omega \neq 0$. Then $e|_N$ is the unit field for the $\mathbf{S}$-equivariant Frobenius manifold structure on N. Indeed, clearly, $e|_N \circ \eta = \eta$ for any section η of Θ_N. On the other hand, we have to check that e is ${}^{\omega}\nabla$-horizontal:

$$ {}^{\omega}\nabla e|_N := \nabla(e|_N \star \omega) \star \omega^{-1} = \nabla(\omega) \star \omega^{-1} = 0, \quad \text{as } \nabla(\omega) = 0. $$

(3) Let us assume that N is chosen so that the Euler vector field $\mathfrak{E}$ is tangent to N. Then $\mathfrak{E}|_N$ is the Euler vector field for the Frobenius manifold structure on N, as $R_0 = \mathfrak{E}\star$ leaves E invariant.

(4) We have ${}^{\omega}g(\xi, \eta) = g(\xi \star \omega, \eta \star \omega)$ for any $\xi, \eta \in \Theta_N$.

Remark 2.3.3. Given an R-basis $\boldsymbol{e}^o$ of $E/(t_0, \dots, t_{m-1})E$, there exists a unique system of flat coordinates (t_i) on N for which $\partial_{t_i} \star \omega \equiv e^o_i \bmod (t_0, \dots, t_{m-1})E$. Given any other formal smooth subscheme N' over R satisfying the properties in Proposition 2.3.2, with corresponding coordinates (t'_i), we do not know whether the natural isomorphism $\mathcal{O}_N \to \mathcal{O}_{N'}$, $t_i \mapsto t'_i$, is compatible with the $\mathbf{S}$-equivariant Frobenius structures. In other words, there is a priori no uniqueness in the construction resulting from Proposition 2.3.2. However, when this construction is applied to the setting of Sect. 3.1, Conjecture 3.7.1 also gives uniqueness.

3. The abelian/nonabelian correspondence for Frobenius structures

A precise relation between the genus zero Gromov–Witten theory (with descendants) of a quotient by a nonabelian group and a twist of the theory for the quotient by a maximal torus in the group was conjectured in [BCK2]. Here we formulate a version of this correspondence for the associated Frobenius structures.

3.1. Setting. Let X be a smooth projective variety over $\mathbb{C}$ with the (linearized) action of a complex reductive group $\mathbf{G}$, and let $\mathbf{T} \subset \mathbf{G}$ be a maximal torus. In this setting, there are two geometric invariant theory (GIT) quotients, $X/\!/\mathbf{T}$ and $X/\!/\mathbf{G}$. We assume (for both actions) that all semistable points are stable and that all isotropy groups of stable points are trivial, so that $X/\!/\mathbf{T}$ and $X/\!/\mathbf{G}$ are smooth projective varieties. Further, we assume that the $\mathbf{G}$-unstable locus $X \setminus X^s(\mathbf{G})$ has codimension at least 2 in X. (Note that this last condition is automatic when X is a projective space.)

There is a diagram

$$
\begin{array}{ccc}
X/\!/\mathbf{T} = X^s(\mathbf{T})/\mathbf{T} & \xleftarrow[j]{} & X^s(\mathbf{G})/\mathbf{T} \\
& & \Big\downarrow{\scriptstyle \pi} \\
& & X/\!/\mathbf{G} = X^s(\mathbf{G})/\mathbf{G}
\end{array}
$$

with j an open immersion and π a $\mathbf{G}/\mathbf{T}$-fibration.

The above diagram leads to a comparison of the cohomology of the non-abelian quotient $X//\mathbf{G}$ to that of the abelian quotient $X//\mathbf{T}$ [ES,Mar,Kir]. We describe an equivariant version of it. Let another (possibly trivial) complex torus $\mathbf{S}$ act on X. Assume that the action commutes with the action of $\mathbf{G}$ and preserves $X^s(\mathbf{T})$ and $X^s(\mathbf{G})$. There is an induced action of $\mathbf{S}$ on the smooth projective varieties $X//\mathbf{T}$ and $X//\mathbf{G}$. The morphisms in the diagram are $\mathbf{S}$-equivariant. To the pair $(\mathbf{G}, \mathbf{T})$ we associate the usual Lie-theoretic data:

- the Weyl group $\mathbf{W} = N(\mathbf{T})/\mathbf{T}$ ($N(\mathbf{T})$ is the normalizer of $\mathbf{T}$ in $\mathbf{G}$).
- the root system Φ with decomposition $\Phi = \Phi_+ \cup \Phi_-$ into positive and negative roots.
- for each root $\alpha \in \Phi$ the 1-dimensional $\mathbf{T}$-representation $\mathbb{C}_\alpha$ with weight α.

The Weyl group acts on $X//\mathbf{T}$, hence also on the equivariant cohomology ring $H^*_{\mathbf{S}}(X//\mathbf{T}, \mathbb{C})$. The representations $\mathbb{C}_\alpha$ define $\mathbf{S}$-equivariant line bundles

$$L_\alpha := X^s(\mathbf{T}) \times_{\mathbf{T}} \mathbb{C}_\alpha$$

on $X//\mathbf{T}$ with equivariant first Chern classes $c_1^{\mathbf{S}}(L_\alpha) \in H^*_{\mathbf{S}}(X//\mathbf{T}, \mathbb{C})$. The $\mathbf{S}$-action on L_α is induced by the $\mathbf{S}$-action on $X^s(\mathbf{T})$ (and the trivial $\mathbf{S}$-action on $\mathbb{C}_\alpha$). Note that $L_{-\alpha} \cong L_\alpha^\vee$ for any pair $(\alpha, -\alpha)$ of opposite roots. The equivariant cohomology class

$$\omega := \sqrt{\frac{1}{|\mathbf{W}|} \prod_{\alpha \in \Phi} c_1^{\mathbf{S}}(L_\alpha)} = \sqrt{\frac{(-1)^{|\Phi_+|}}{|\mathbf{W}|}} \prod_{\alpha \in \Phi_+} c_1^{\mathbf{S}}(L_\alpha)$$

will play an important role in this paper. It is the fundamental $\mathbf{W}$-anti-invariant class in the equivariant cohomology of $X//\mathbf{T}$; any other $\mathbf{W}$-anti-invariant class ϕ can be written (non-uniquely) as $\gamma \cup \omega$, with $\gamma \in H^*_{\mathbf{S}}(X//\mathbf{T}, \mathbb{C})^{\mathbf{W}}$. (The reason for considering ω rather than the customary $\Delta = \prod_{\alpha \in \Phi_+} c_1^{\mathbf{S}}(L_\alpha)$ is one of convenience: we simply want to avoid having to insert the factor $(-1)^{|\Phi_+|}/|\mathbf{W}|$ in all formulae comparing Gromov–Witten invariants of $X//\mathbf{G}$ and $X//\mathbf{T}$.)

The following facts are known:

(3.1.1) π^* induces an isomorphism $H^*_{\mathbf{S}}(X//\mathbf{G}) \cong H^*_{\mathbf{S}}(X^s(\mathbf{G})/\mathbf{T})^{\mathbf{W}}$

(3.1.2) There is an exact sequence

$$0 \longrightarrow \ker(\cup\omega) \xrightarrow{\subset} H^*_{\mathbf{S}}(X//\mathbf{T})^{\mathbf{W}} \xrightarrow[(\pi^*)^{-1} \circ j^*]{} H^*_{\mathbf{S}}(X//\mathbf{G}) \longrightarrow 0$$

where $\ker(\cup\omega)$ is $\{\gamma \in H^*_{\mathbf{S}}(X//\mathbf{T})^{\mathbf{W}} \mid \gamma \cup \omega = 0\}$.

(3.1.3) The equivariant push-forwards satisfy the equality

$$\int_{X//\mathbf{T}} \omega^2 \tilde{\sigma} = \int_{X//\mathbf{G}} \sigma$$

for all $\sigma \in H^*_{\mathbf{S}}(X//\mathbf{G})$, $\tilde{\sigma} \in H^*_{\mathbf{S}}(X//\mathbf{T})$ with $j^*\tilde{\sigma} = \pi^*(\sigma)$. (Such $\tilde{\sigma}$ are called *lifts* of σ.)

(3.1.4) There is an identification of the **S**-equivariant relative tangent bundle T_π of $\pi : X^s(\mathbf{G})/\mathbf{T} \to X^s(\mathbf{G})/\mathbf{G}$ with $\oplus_{\alpha\in\Phi} L_\alpha|_{X^s(\mathbf{G})/\mathbf{T}}$.

In the nonequivariant case (that is, for **S**=1), (3.1.1) is a classical result, (3.1.2) is proved in [ES] for $X = \mathbb{P}^N$ and in [Kir] in general, (3.1.3) is proved in [Mar] and (3.1.4) can be seen by a direct computation. The extensions to the equivariant context are straightforward and left to the reader.

3.2. The W-induced Frobenius manifold. Applying the results in Sect. 2.3 to the Weyl group action on the **S**-equivariant Frobenius manifold given by the equivariant Gromov–Witten theory of $X//\mathbf{T}$, a new **S**-equivariant Frobenius manifold (of dimension over the base ring equal to the rank of $H^*_{\mathbf{S}}(X//\mathbf{G}, \mathbb{C})$) is obtained. In this subsection we spell out for concreteness the details of the construction and the main properties of the new Frobenius structure in this special case.

As mentioned in the introduction, a specialization of Novikov variables will be needed before comparing the new Frobenius structure with the one given by the equivariant Gromov–Witten theory of $X//\mathbf{G}$ and we start with this specialization.

Recall from (3.1.3) the notion of lift of cohomology classes from $X//\mathbf{G}$ to $X//\mathbf{T}$. By (3.1.2), one can always choose **W**-invariant lifts. These are not generally unique; however, the assumption that the **G**-unstable locus in X has codimension ≥ 2 implies that for divisor classes the **W**-invariant lifts are unique. This allows us to lift curve classes as well (cf. [BCK2]): the inclusion

$$\mathrm{Pic}(X//\mathbf{G}) \cong \mathrm{Pic}(X//\mathbf{T})^{\mathbf{W}} \subset \mathrm{Pic}(X//\mathbf{T})$$

induces by duality a projection

$$\varrho : NE_1(X//\mathbf{T}) \longrightarrow NE_1(X//\mathbf{G}).$$

We say that $\tilde{\beta}$ lifts $\beta \in NE_1(X//\mathbf{G})$ (and write $\tilde{\beta} \mapsto \beta$) if $\varrho(\tilde{\beta}) = \beta$. Note that any effective β has finitely many lifts. Define a projection on Novikov rings

(3.2.1)
$$p : N(X//\mathbf{T}) \to N(X//\mathbf{G}), \quad p\Big(\sum_{\tilde{\beta}} c_{\tilde{\beta}}\, Q^{\tilde{\beta}}\Big) = \sum_{\beta} (-1)^{\epsilon(\beta)} \Big(\sum_{\tilde{\beta}\mapsto\beta} c_{\tilde{\beta}}\Big) Q^{\beta},$$

where

$$\epsilon : NE_1(X//\mathbf{G}) \longrightarrow \mathbb{Z}_2$$

is defined by

$$\epsilon(\beta) = \Big(\int_{\tilde{\beta}} \sum_{\alpha\in\Phi_+} c_1^{\mathbf{S}}(L_\alpha)\Big) \pmod 2$$

with $\tilde{\beta}$ any lift of β. This makes sense, since the right-hand side does not depend on the choice of lift. Indeed, if α' is any simple root and $\upsilon_{\alpha'} \in \mathbf{W}$ is the corresponding reflection, then by standard properties of root systems

$$\upsilon_{\alpha'}\Big(\sum_{\alpha\in\Phi_+} c_1^{\mathbf{S}}(L_\alpha)\Big) = \sum_{\alpha\in\Phi_+} c_1^{\mathbf{S}}(L_\alpha) - 2c_1^{\mathbf{S}}(L_{\alpha'}),$$

so $\sum_{\alpha\in\Phi_+} c_1^{\mathbf{S}}(L_\alpha)$ is $\mathbf{W}$-invariant as a cohomology class with $\mathbb{Z}_2$-coefficients.

The sign in (3.2.1), which may seem rather mysterious, has its origin in the twisting bundle appearing in the abelian/nonabelian correspondence, as formulated in [BCK2, Conjecture 4.2].

Let Z be the formal Frobenius manifold defined by the $\mathbf{S}$-equivariant Gromov–Witten theory of $X//\mathbf{T}$, with potential function $F^{X//\mathbf{T},\mathbf{S}}$. Let M be the formal scheme over $N(X//\mathbf{G}) \otimes \mathbb{C}[\lambda]$ obtained by base change from Z by the morphism of Novikov rings (3.2.1). Let $\theta : M \longrightarrow Z$ be the base change map. We obtain a formal Frobenius structure over $N(X//\mathbf{G}) \otimes \mathbb{C}[\lambda]$ on (M, Θ_M) by pulling-back via θ the Frobenius structure on Z. Note that only the potential (and therefore the quantum product) changes under the pull-back, since the coefficients of the metric, the horizontal sections and the Euler vector field do not depend on the Novikov variables. Explicitly, the potential of the Frobenius structure on M is

(3.2.2)
$$F := \theta^*(F^{X//\mathbf{T},\mathbf{S}}) = \sum_{\beta\in NE_1(X//\mathbf{G})} (-1)^{\epsilon(\beta)} Q^\beta \sum_{n\geq 0} \frac{1}{n!}\Big(\sum_{\tilde{\beta}\mapsto\beta} \langle \underbrace{\gamma,\ldots,\gamma}_{n} \rangle_{0,n,\tilde{\beta}}^{X//\mathbf{T},\mathbf{S}} \Big).$$

Choose a homogeneous basis $\{\sigma_0 = 1, \sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_{m-1}\}$ of $H^*_{\mathbf{S}}(X//\mathbf{G})$ over $\mathbb{C}[\lambda] := \mathbb{C}[\lambda_1, \ldots, \lambda_\ell] = H^*(B\mathbf{S})$, such that $\{\sigma_1, \ldots, \sigma_r\}$ forms a basis of $H^2_{\mathbf{S}}(X//\mathbf{G})$ and *fix* homogeneous lifts $\gamma_i \in H^*_{\mathbf{S}}(X//\mathbf{T})^{\mathbf{W}}$ of σ_i. The fixed lifts give rise to a $\mathbb{C}$-linear embedding

(3.2.3)
$$H^*_{\mathbf{S}}(X//\mathbf{G}, \mathbb{C}) \subset H^*_{\mathbf{S}}(X//\mathbf{T}, \mathbb{C})$$

(which may not in general be a homomorphism of equivariant cohomology rings).

The image of the embedding (3.2.3) determines a formal submanifold N of M over $N(X//\mathbf{T}) \otimes \mathbb{C}[\lambda]$.

Let

$$V := H^*_{\mathbf{S}}(X//\mathbf{T}, \mathbb{C})^a$$

be the subspace of $\mathbf{W}$-anti-invariant classes. The composition of (3.2.3) with the map

$$\cup\omega : H^*_{\mathbf{S}}(X//\mathbf{T}, \mathbb{C})^{\mathbf{W}} \to H^*_{\mathbf{S}}(X//\mathbf{T}, \mathbb{C})^a$$

is an isomorphism from $H^*_{\mathbf{S}}(X//\mathbf{G}, \mathbb{C})$ to V. Let $\mathcal{V} = V \otimes \mathcal{O}_N$ be the subsheaf of $\Theta_M|_N$ induced by V. Let $\star$ be the quantum product on Θ_M (that is, the pull-back by θ of the quantum product on $H^*_{\mathbf{S}}(X//\mathbf{T}, \mathbb{C})$). Consider the map

$$\star\,\omega : (\Theta_M|_N)^{\mathbf{W}} \longrightarrow \mathcal{V}, \quad \xi \mapsto (\hat{\xi} \star \omega)|_N,$$

with $\hat{\xi} \in \Theta_M^{\mathbf{W}}$ any extension of ξ to M. (It is well defined, since the quantum product of two vector fields at a point depends only on their values at the point.) This map reduces to $\cup\omega$ modulo the ideal generated by $\{Q^\beta | \beta \neq 0\}$. By Nakayama's lemma, $\star\,\omega$ induces an isomorphism $\Theta_N \to \mathcal{V}$. Let $\phi : \mathcal{V} \to \Theta_N$ be the inverse isomorphism. Abusing notation, when $\eta \in \mathcal{V}$ we write $\eta \star \omega^{-1}$ for $\phi(\eta)$. Hence we have for $\xi \in \Theta_N$

$$(\xi \star \omega) \star \omega^{-1} = \xi.$$

We now induce a structure of formal Frobenius manifold on N (over $N(X//\mathbf{G}) \otimes \mathbb{C}[\lambda]$) using the maps $\star\,\omega$ and $\star\,\omega^{-1}$. Explicitly:

(3.2.4) The metric ${}^\omega g$ on Θ_N is given by the composition

$$\Theta_N \otimes \Theta_N \hookrightarrow \Theta_M|_N \otimes \Theta_M|_N \xrightarrow{\star\,\omega \otimes \star\,\omega} \mathcal{V} \otimes \mathcal{V} \xrightarrow{g|_{\mathcal{V}}} \mathcal{O}_N,$$

that is,

$${}^\omega g(\xi, \eta) = g|_{\mathcal{V}}(\xi \star \omega, \eta \star \omega).$$

Note that $g|_{\mathcal{V}}$ is nondegenerate on $\mathcal{V}$ by Martin's formula (3.1.3).

(3.2.5) The Levi–Civita connection ${}^\omega\nabla$ of ${}^\omega g$ satisfies

$${}^\omega\nabla_\xi \eta = (\nabla_{\hat{\xi}}(\hat{\eta} \star \omega))|_N \star \omega^{-1}.$$

(3.2.6) The product of $\xi, \eta \in \Theta_N$ is defined by

$$\xi \circ \eta = (\xi \star \eta \star \omega) \star \omega^{-1}.$$

In other words, $\xi \circ \eta$ is the projection of $\xi \star \eta$ along $\ker(\star\,\omega)$.

(3.2.7) The unit is the vector field 1 restricted to N.

The symmetry of ${}^\omega\nabla({}^\omega g(\cdot \circ \cdot, \cdot))$ and the corresponding potential function are discussed in Sect. 3.5 below.

3.3. Flat coordinates. On N there are coordinates $\tilde{t}_0, \ldots, \tilde{t}_{m-1}$ determined by the basis $\{\gamma_0 = 1, \gamma_1, \ldots, \gamma_r, \ldots, \gamma_{m-1}\}$ of lifts introduced above. These are just restrictions to N of coordinates on M which are flat for the connection ∇. Let

(3.3.1) $$\xi_i(\tilde{t}) := (\gamma_i \cup \omega) \star \omega^{-1}, \quad i = 0, \ldots, m-1.$$

Equivalently, ξ_i is defined by the equality

(3.3.2) $$\xi_i \star \omega = \gamma_i \cup \omega.$$

The ξ_i's form a basis of Θ_N consisting of ${}^\omega\nabla$-horizontal vector fields. Denote by $s := (s_0, s_1, \ldots, s_r, \ldots, s_{m-1})$ the corresponding ${}^\omega\nabla$-flat coordinates on N (so that $\partial_{s_i} = \xi_i$). Note that

(3.3.3) $$s \equiv \tilde{t}, \text{ modulo the ideal generated by } \{Q^\beta | \beta \neq 0\}.$$

3.4. Euler vector field. Since

$$\begin{aligned} j^* c_1^{\mathbf{S}}(T(X//\mathbf{T})) &= c_1^{\mathbf{S}}(T(X^s(\mathbf{G})/\mathbf{T})) \\ &= \pi^*\big(c_1^{\mathbf{S}}(T(X//\mathbf{G}))\big) + \sum_{\alpha\in\Phi} c_1^{\mathbf{S}}(L_\alpha) \\ &= \pi^*\big(c_1^{\mathbf{S}}(T(X//\mathbf{G}))\big) \end{aligned}$$

and $\mathrm{Pic}_{\mathbf{S}}(X//\mathbf{T}) \cong \mathrm{Pic}_{\mathbf{S}}(X^s(\mathbf{G})/\mathbf{T})$ via j^*, we conclude that $c_1^{\mathbf{S}}(T(X//\mathbf{T}))$ is $\mathbf{W}$-invariant. Viewing $c_1^{\mathbf{S}}(T(X//\mathbf{T}))$ as a vector field on M, its restriction to N is therefore a section of $(\Theta_M|_N)^{\mathbf{W}}$. Moreover, this restriction is in fact tangent to N, since (by uniqueness of lifts of divisors) N contains the germ of linear subspace $H^2_{\mathbf{S}}(X//\mathbf{T})^{\mathbf{W}}$.

Define the Euler vector field by

$$\mathfrak{E}_{\mathbf{S}} = \mathfrak{E} + \sum_{i=1}^{\ell} \lambda_i \partial_{\lambda_i}$$

with

$$\mathfrak{E} = \sum_{i=0}^{m-1} \left(1 - \frac{\deg \sigma_i}{2}\right) \tilde{t}_i \partial_{\tilde{t}_i} + c_1^{\mathbf{S}}(T(X//\mathbf{T}))|_N .$$

Note that $\mathfrak{E}_{\mathbf{S}}$ is simply the restriction to N of the corresponding Euler vector field for $X//\mathbf{T}$ (see Sect. 2.2): Extend $\{\gamma_0, \ldots, \gamma_{m-1}\}$ to a basis of $H^*_{\mathbf{S}}(X//\mathbf{T})$ for the Euler vector field for $X//\mathbf{T}$.

Applying $\mathcal{L}_{\mathfrak{E}_{\mathbf{S}}}$ to the equality $(\xi_i \star \omega) = \partial_{\tilde{t}_i} \cup \omega$, we see that

$$\mathcal{L}_{\mathfrak{E}_{\mathbf{S}}} \xi_i = \left(1 - \frac{\deg \sigma_i}{2}\right) \xi_i .$$

Easy calculations show then that

$$\mathcal{L}_{\mathfrak{E}_{\mathbf{S}}}({}^{\omega}g) = (2 - \dim(X//\mathbf{G}))\, {}^{\omega}g, \qquad \mathcal{L}_{\mathfrak{E}_{\mathbf{S}}}(\circ) = \circ$$

(and obviously $\mathcal{L}_{\mathfrak{E}_{\mathbf{S}}}(1) = -1$), hence $\mathfrak{E}_{\mathbf{S}}$ is indeed an Euler vector field. Also,

$$\mathcal{L}_{\mathfrak{E}_{\mathbf{S}}}(s_i) = \deg(\tilde{t}_i) s_i ,$$

that is, $\deg s_i = \deg \tilde{t}_i$. In particular, $\deg s_1 = \cdots = \deg s_r = 0$.

3.5. Potential. Recall that we identify the cohomology classes on $X//\mathbf{T}$ with $\mathcal{O}_M$-linear vector fields on M. Denote by $\partial_{\tilde{t}_i \cup \omega}$ the vector field corresponding to $\gamma_i \cup \omega$. The components of the tensor "$\circ$" in the basis of ${}^{\omega}\nabla$-horizontal fields are

(3.5.1)
$$ {}^{\omega}g(\xi_i \circ \xi_j, \xi_k) = g|_V(\xi_i \star \xi_j \star \omega, \xi_k \star \omega) = g(\hat{\xi}_i \star (\gamma_j \cup \omega), \gamma_k \cup \omega)|_N $$

where $\hat{\xi}_i$ is any extension of ξ to a **W**-invariant vector field on M. Since

$$ \begin{aligned} g(\hat{\xi}_i \star (\gamma_j \cup \omega), \gamma_k \cup \omega)|_N &= g(\gamma_j \cup \omega, \hat{\xi}_i \star (\gamma_k \cup \omega))|_N \\ &= g|_V(\gamma_j \star \omega, \xi_i \star \xi_k \star \omega) = {}^{\omega}g(\xi_j, \xi_i \circ \xi_k), \end{aligned} $$

we see that the Frobenius algebra property

$$ {}^{\omega}g(\xi_i \circ \xi_j, \xi_k) = {}^{\omega}g(\xi_j, \xi_i \circ \xi_k) \tag{3.5.2} $$

holds. Recall the potential F (see (3.2.2)) of the formal Frobenius manifold M. We get from (3.5.1)

$$ \begin{aligned} {}^{\omega}g(\xi_i \circ \xi_j, \xi_k) &= g(\hat{\xi}_i \star (\gamma_j \cup \omega), \gamma_k \cup \omega)|_N \\ &= (\hat{\xi}_i(\partial_{\tilde{t}_j \cup \omega}\partial_{\tilde{t}_k \cup \omega} F))|_N = \xi_i((\partial_{\tilde{t}_j \cup \omega}\partial_{\tilde{t}_k \cup \omega} F)|_N). \end{aligned} \tag{3.5.3} $$

Note that

$$ \xi_l(\xi_i((\partial_{\tilde{t}_j \cup \omega}\partial_{\tilde{t}_k \cup \omega} F)|_N)) = \xi_i(\xi_l((\partial_{\tilde{t}_j \cup \omega}\partial_{\tilde{t}_k \cup \omega} F)|_N)), $$

since $[\xi_l, \xi_i] = 0$. Hence

$$ \xi_l\big({}^{\omega}g(\xi_i \circ \xi_j, \xi_k)\big) = \xi_i\big({}^{\omega}g(\xi_l \circ \xi_j, \xi_k)\big). $$

Combined with (3.5.2), this implies that the tensor $\xi_l({}^{\omega}g(\xi_i \circ \xi_j, \xi_k))$ is symmetric in the indices l, i, j, k, hence there is a (formal) function F' on N such that

$$ \partial_{s_i}\partial_{s_j}\partial_{s_k} F' = {}^{\omega}g(\xi_i \circ \xi_j, \xi_k). $$

Finally, a direct computation shows that $L_{\mathfrak{E}_{\mathbf{S}}} F' = (3 - \dim X//\mathbf{G}) F'$ up to quadratic terms.

This finishes the construction of the induced structure of formal **S**-equivariant Frobenius manifold over $N(X//\mathbf{G}) \otimes \mathbb{C}[\lambda]$ on N.

3.6. More on the flat coordinates. For later use we record here some details about the change of coordinates $s_i(\tilde{t})$ on N. From the defining equation (3.3.1) for the horizontal vector fields ξ_i it follows that the jacobian matrix $A := (\partial s_i/\partial \tilde{t}_j)_{i,j}$ is given explicitly by

$$\frac{\partial s_i}{\partial \tilde{t}_j} = \left(\partial_{\tilde{t}_j}\partial^{\vee}_{\tilde{t}_i\cup\omega}\partial_{\tilde{t}_0\cup\omega}F\right)\Big|_N$$

where

$$\partial^{\vee}_{\tilde{t}_i\cup\omega} := \sum_k g^{ik}\partial_{\tilde{t}_k\cup\omega}$$

with $(g^{ik}) \in GL_m(\mathbb{C}[\lambda])$ the inverse matrix of the metric $g|_{\mathcal{V}}$. Using the divisor axiom for Gromov–Witten invariants of $X//\mathbf{T}$ in the formula (3.2.2) for the potential F, we see that the entries of the jacobian matrix have the form

$$\begin{aligned}(3.6.1)\quad &\frac{\partial s_i}{\partial \tilde{t}_0} = \delta_{i0}\\ &\frac{\partial s_i}{\partial \tilde{t}_j} = \delta_{ij} + \sum_{\beta\neq 0} Q^{\beta}e^{\beta\cdot\tilde{t}_{\text{small}}}c_{\beta,ij}(\tilde{t}_{r+1},\ldots,\tilde{t}_{m-1}), \quad j\neq 0,\end{aligned}$$

where $c_{\beta,ij} \in \mathbb{C}[\lambda][[\tilde{t}_{r+1},\ldots,\tilde{t}_{m-1}]]$ and

$$\beta\cdot\tilde{t}_{\text{small}} = \sum_{i=1}^{r}\tilde{t}_i\int_{\beta}\sigma_i.$$

By integrating (3.6.1) (with the initial condition $s(0) = 0$), we obtain a refined version of (3.3.3)

$$(3.6.2)\qquad s_i = \tilde{t}_i + \sum_{\beta\neq 0} Q^{\beta}e^{\beta\cdot\tilde{t}_{\text{small}}}b_{\beta,i}(\tilde{t}_{r+1},\ldots,\tilde{t}_{m-1}),$$

with $b_{\beta,i} \in \mathbb{C}[\lambda][[\tilde{t}_{r+1},\ldots,\tilde{t}_{m-1}]]$.

By considering the inverse jacobian matrix (which gives the map $\star\,\omega^{-1}$), it follows that the inverse coordinate change $\tilde{t}(s)$ is also of the type (3.6.2), hence the potential function F' in flat coordinates s_i has the special form (2.1.1) (up to quadratic terms)

$$(3.6.3)\qquad F' = F'_{cl} + \sum_{\beta\neq 0} Q^{\beta}e^{\beta\cdot s_{\text{small}}}F'_{\beta}(s_{r+1},\ldots,s_{m-1})$$

where $\beta\cdot s_{\text{small}} = \sum_{i=1}^{r} s_i(\int_{\beta}\sigma_i)$.

Finally, we record what happens with the "small" parameter spaces under the change of coordinates.

Lemma 3.6.1. (*i*) *If $X//\mathbf{G}$ is Fano of index ≥ 2, then the subspaces of N given by the equations $\{s_0 = s_{r+1} = \cdots = s_{m-1} = 0\}$ and $\{\tilde{t}_0 = \tilde{t}_{r+1} = \cdots = \tilde{t}_{m-1} = 0\}$ coincide. Moreover, on this subspace we have $s_i = \tilde{t}_i$ for $i = 1,\ldots,r$.*

(ii) If $c_1(T(X//\mathbf{G}))$ *is nef, then the subspaces* $\{s_{r+1} = \cdots = s_{m-1} = 0\}$ *and* $\{\tilde{t}_{r+1} = \cdots = \tilde{t}_{m-1} = 0\}$ *coincide.*

Proof. (i) Let $1 \le i \le r$. After restriction to $\tilde{t}_0 = \tilde{t}_{r+1} = \cdots = \tilde{t}_{m-1} = 0$ we obtain

$$\xi_i = \gamma_i + \sum_j \Big(\sum_\beta c_{\beta,ij} Q^\beta e^{\beta\cdot\tilde{t}_{\mathrm{small}}} \Big) \gamma_j .$$

Since $\deg \xi_i = 1$ and $\deg e^{\beta\cdot\tilde{t}_{\mathrm{small}}} = \int_\beta c_1(T(X//\mathbf{G})) \ge 2$, we must have $\xi = \gamma_i$ and the statement follows. The proof of (ii) is similar. □

3.7. Main conjecture. Let P be the formal $\mathbf{S}$-equivariant Frobenius manifold over $N(X//\mathbf{G}) \otimes \mathbb{C}[\lambda]$ defined by the genus zero $\mathbf{S}$-equivariant Gromov–Witten theory of $X//\mathbf{G}$, with flat coordinates $t_0, t_1, \ldots, t_r, \ldots, t_{m-1}$ corresponding to the $\mathbb{C}[\lambda]$-basis $\{\sigma_0 = 1, \sigma_1, \ldots, \sigma_r, \ldots, \sigma_{m-1}\}$ of $H^*_{\mathbf{S}}(X//\mathbf{G})$ and potential function $F^{X//\mathbf{G},\mathbf{S}}$. We are now ready to formulate the abelian/nonabelian correspondence:

Conjecture 3.7.1. Let $\varphi : P \longrightarrow N$ be the isomorphism of formal schemes over $N(X//\mathbf{G}) \otimes \mathbb{C}[\lambda]$ defined by $\varphi^*(s_i) = t_i$. Then φ induces an isomorphism of formal $\mathbf{S}$-equivariant Frobenius structures such that $\varphi^*(\xi_i) = \sigma_i$ and $\varphi^* F' = F^{X//\mathbf{G},\mathbf{S}}$ up to quadratic terms.

Note that $\varphi^*(\xi_i) = \sigma_i$ follows easily from (3.3.1). The main point of the conjecture is the identification of potentials. We also remark that the conjecture implies in particular that the new $\mathbf{W}$-induced Frobenius structure constructed in this section does not depend on the choice of the $\mathbf{W}$-invariant lift of $H^*_{\mathbf{S}}(X//\mathbf{G}, \mathbb{C})$.

4. Proof of Conjecture 3.7.1 for flag manifolds

4.1. Preliminaries. Let $0 < k_1 < \cdots < k_r < n = k_{r+1}$ be integers. Consider the vector space

$$\Omega := \bigoplus_{i=1}^{r} \mathrm{Mat}_{k_i \times k_{i+1}}(\mathbb{C})$$

where $\mathrm{Mat}_{k_i \times k_{i+1}}(\mathbb{C})$ is the space of matrices of size $k_i \times k_{i+1}$ with complex entries. Let $\mathbf{G} := \prod_{i=1}^r GL_{k_i}(\mathbb{C})$, with maximal torus $\mathbf{T}$ equal to the product of the subgroups of diagonal matrices. $\mathbf{G}$ acts on Ω by

$$(g_1, \ldots, g_r)(A_1, \ldots, A_r) = \big(g_1 A_1 g_2^{-1}, g_2 A_2 g_3^{-1}, \ldots, g_{r-1} A_{r-1} g_r^{-1}, g_r A_r\big).$$

This action descends to an action on $X := \mathbb{P}(\Omega)$, with a canonical linearization on $\mathcal{O}(1)$, and the GIT quotient $X//\mathbf{G}$ is the partial flag manifold $Fl(k_1, \ldots, k_r, n)$ parameterizing flags of subspaces $\{\mathbb{C}^{k_1} \subset \cdots \subset \mathbb{C}^{k_r} \subset \mathbb{C}^n\}$.

The corresponding abelian quotient $X//\mathbf{T}$ is a toric variety which can be realized as a tower of fibered products of projective bundles.

Let $\mathbf{S} \cong (\mathbb{C}^*)^n$ be the subgroup of diagonal matrices in $GL_n(\mathbb{C})$, acting on Ω by right-multiplication of A_r. There are induced $\mathbf{S}$-actions on $X//\mathbf{G}$ (which is just the usual action of the maximal torus in GL_n on the flag manifold) and on $X//\mathbf{T}$. See [BCK2, §5.1] for more details on $X//\mathbf{G}$, $X//\mathbf{T}$, and the $\mathbf{S}$-actions on them. As before, we let $\mathbb{C}[\lambda] = \mathbb{C}[\lambda_1, \ldots, \lambda_n] = H^*(B\mathbf{S}, \mathbb{C})$, with quotient field $\mathbb{C}(\lambda)$. Our goal in this section is to prove

Theorem 4.1.1. *Conjecture 3.7.1 holds for*

(*a*) *the usual Gromov–Witten theory of* $Fl(k_1, \ldots, k_r, n)$.

(*b*) *the* $\mathbf{S}$-*equivariant Gromov–Witten theory of* $Fl(k_1, \ldots, k_r, n)$.

Remark 4.1.2. Note that part (*a*) follows from (*b*) by taking the non-equivariant limit $\lambda_1 = \cdots = \lambda_n = 0$ of the potential functions.

Our strategy for proving Theorem 4.1.1 is to use reconstruction theorems to reduce the statement to a comparison for 1-point invariants which was established in [BCK1,BCK2]. Typically, reconstruction results for Gromov–Witten invariants work under the assumption that the cohomology ring is generated by divisors. Our observation here is that in the torus-equivariant setting, this assumption needs only to hold after localization. This enlarges the class of varieties for which reconstruction is applicable. We begin with a simple lemma.

Lemma 4.1.3. *Let* $\mathbb{P}^N$ *be acted by a torus* $\mathbf{S}$ *and let* Y *be an* $\mathbf{S}$-*invariant smooth subvariety. Suppose that the natural map* $H^*((\mathbb{P}^N)^{\mathbf{S}}) \to H^*(Y^{\mathbf{S}})$ *is surjective (for example, this is true when the* $\mathbf{S}$-*fixed locus* $(\mathbb{P}^N)^{\mathbf{S}}$ *is isolated). Then the localized equivariant cohomology ring* $H^*_{\mathbf{S}}(Y, \mathbb{C}) \otimes_{\mathbb{C}[\lambda]} \mathbb{C}(\lambda)$ *is generated (as a* $\mathbb{C}(\lambda)$-*algebra) by divisors, i.e., by* $\{c_1(L) \otimes 1 | L \in \mathrm{Pic}^{\mathbf{S}}(Y)\}$.

Proof. There is a commutative diagram

$$\begin{array}{ccc} H^*_{\mathbf{S}}(\mathbb{P}^N) & \longleftarrow & H^*((\mathbb{P}^N)^{\mathbf{S}}) \otimes \mathbb{C}[\lambda] \\ \downarrow & & \downarrow \\ H^*_{\mathbf{S}}(Y) & \longleftarrow & H^*(Y^{\mathbf{S}}) \otimes \mathbb{C}[\lambda]. \end{array}$$

After tensor product with $\mathbb{C}(\lambda)$, the horizontal maps are isomorphism by the localization theorem. □

It is well-known that $X//\mathbf{G} = Fl(k_1, \ldots, k_r, n)$ admits an $\mathbf{S}$-equivariant embedding into a product of projective spaces on which $\mathbf{S}$ acts with isolated fixed points. By Lemma 4.1.3, the localized equivariant cohomology

$$H^*_{\mathbf{S}}(X//\mathbf{G}, \mathbb{C}) \otimes_{\mathbb{C}[\lambda]} \mathbb{C}(\lambda)$$

is generated by divisor classes. Note that this is false in general without localization. For example the equivariant cohomology of Grassmannians is *not* generated by divisors. (On the other hand, since $X//\mathbf{T}$ is a toric variety, both the usual and $\mathbf{S}$-equivariant cohomology rings are already generated by divisors.)

Before going into the details of the proof, it is useful to discuss the base-change of Novikov rings (3.2.1) in the particular case of flag manifolds. By choosing the usual Schubert basis in $H_2(Fl(k_1, \ldots, k_r, n), \mathbb{Z})$, the semigroup of effective curve classes is identified with $(\mathbb{Z}_+)^r$. We write $d = (d_1, \ldots, d_r)$ for the typical element in this semigroup. The Novikov ring is simply the power series ring $\mathbb{C}[[Q_1, \ldots, Q_r]]$. Similarly, effective curve classes on the toric variety $X//\mathbf{T}$ are described by tuples of non-negative integers

$$\tilde{d} = (d_{11}, \ldots, d_{1k_1}, \ldots, d_{r1}, \ldots, d_{rk_r})$$

and the Novikov ring is identified with $\mathbb{C}[[Q_{ij} | 1 \leq i \leq r, 1 \leq j \leq k_i]]$. A class $\tilde{d}$ is a lift of d if and only if

$$d_i = \sum_{j=1}^{k_i} d_{ij}, \quad i = 1, \ldots, r.$$

Finally, $\epsilon(d) = \sum_{i=1}^{r} (k_i - 1) d_i \pmod 2$. Hence the projection (3.2.1) of Novikov rings is

$$\text{(4.1.1)} \quad \begin{aligned} p : \mathbb{C}[[Q_{ij} | 1 \leq i \leq r,\ 1 \leq j \leq k_i]] &\longrightarrow \mathbb{C}[[Q_1, \ldots, Q_r]], \\ Q_{ij} &\longmapsto (-1)^{(k_i-1)} Q_i. \end{aligned}$$

4.2. Kontsevich–Manin reconstruction and reduction to 2-point invariants. This step involves an equivariant version of the Kontsevich–Manin reconstruction theorem. In its original formulation [KM], the reconstruction theorem states that if the cohomology ring $H^*(Y, \mathbb{C})$ is generated by divisors, then all Gromov–Witten invariants of Y can be reconstructed from 3-point invariants for which at least one insertion is a divisor class. These in turn are expressed in terms of 2-point invariants by using the divisor equation in Gromov–Witten theory. We give here an extension of reconstruction to the $\mathbf{S}$-equivariant setting.

Lemma 4.2.1. *Let Y be a smooth complex projective variety with $\mathbf{S}$-action. Let*

$$P := Spf\big((N(Y) \otimes \mathbb{C}[\lambda])\big[\big[H^*_{\mathbf{S}}(Y, \mathbb{C})^\vee\big]\big]\big),$$

endowed with the formal $\mathbf{S}$-equivariant Frobenius structure $(P, \star, g, 1, \mathfrak{E}_{\mathbf{S}})$ defined by the equivariant Gromov–Witten potential F^Y. Let $t = (t_0, t_1, \ldots, t_r, \ldots, t_{m-1})$ be the flat coordinates defined by a basis of $H^*_{\mathbf{S}}(Y, \mathbb{C})$, such *that* $t_{\text{small}} = (t_1, \ldots, t_r)$ *are the coordinates on the small parameter space* $H^2_{\mathbf{S}}(Y, \mathbb{C})$. *Let* $G \in \mathcal{O}_P$ *be another formal function satisfying the WDVV equations. Assume that:*

(*i*) *In flat coordinates* G *has the form* (2.1.1)

$$G = G_{cl} + \sum_{\beta \in E, \beta \neq 0} Q^{\beta} e^{\beta \cdot t_{\text{small}}} G_{\beta},$$

with $G_\beta \in \mathbb{C}[\lambda][[t_{r+1}, \ldots, t_{m-1}]]$ *and* G_{cl} *a cubic polynomial in the* t_i*'s (with coefficients in* $\mathbb{C}[\lambda]$*).*

(*ii*) $\mathcal{L}_{\mathfrak{E}_{\mathbf{S}}}(G) = (3 - \dim(Y))G$.

(*iii*) $G_{cl} = F^Y_{cl}$.

(*iv*) $\partial_{t_i} \partial_{t_j} G|_{t_{\text{small}}} = \partial_{t_i} \partial_{t_j} F^Y|_{t_{\text{small}}}$, *for all* i, j.

(*v*) *The localized equivariant cohomology ring* $H^*_{\mathbf{S}}(Y, \mathbb{C}) \otimes_{\mathbb{C}[\lambda]} \mathbb{C}(\lambda)$ *is generated by* $H^2_{\mathbf{S}}(Y, \mathbb{C})$ *as a* $\mathbb{C}(\lambda)$*-algebra.*

Then $G = F^Y$.

Proof. Let $P_{(\lambda)}$ be the **S**-equivariant Frobenius manifold defined by the localized Gromov–Witten theory of Y (see Sect. 2.1). The function G defines a formal **S**-equivariant Frobenius structure $(P, \circ, g, 1, \mathfrak{E}_{\mathbf{S}})$ over $N(Y) \otimes \mathbb{C}[\lambda]$, and a localized Frobenius structure over $N(Y) \otimes \mathbb{C}(\lambda)$ as well, by viewing it as a formal function on $P_{(\lambda)}$ via the natural (injective!) localization map $\iota : \mathcal{O}_P \to \mathcal{O}_{P_{(\lambda)}}$. It suffices to check that the localized potentials $F^Y_{(\lambda)} = \iota(F^Y)$ and $G_{(\lambda)} = \iota(G)$ are equal. The assumptions (*i*)–(*iii*) hold for the localized potentials as well (where in (*i*) we replace $\mathbb{C}[\lambda]$ by $\mathbb{C}(\lambda)$).

In the conformal case, a formal Frobenius structure satisfying (*i*) and (*ii*) is said to be of *qc-type* in [Man]. Such structure has "cup product", defined by

$$G_{cl} = \frac{1}{6} g\Big(\Big(\sum t_i \partial_{t_i}\Big) \cup \Big(\sum t_i \partial_{t_i}\Big), \sum t_i \partial_{t_i}\Big)$$

and "correlators"

$$I_{0,n,\beta}(\partial_{t_{i_1}}, \ldots, \partial_{t_{i_n}}) = \partial_{t_{i_1}} \ldots \partial_{t_{i_n}} G_\beta|_{t=0}$$

which satisfy the analogue of the divisor axiom in Gromov–Witten theory. See [Man, § 5.4]. The same will hold for the Frobenius structure defined by our potential G, or for its localized version. We may call them Frobenius structures of equivariant qc-type.

Assumption (*v*) and the usual Kontsevich–Manin reconstruction imply that the localized GW-potential $F^Y_{(\lambda)}$ is determined recursively by $\partial_{t_i} \partial_{t_j} F^Y_{(\lambda)}|_{t_{\text{small}}}$. The proof only uses properties of Gromov–Witten invariants which are shared by the correlators of any Frobenius structure of equivariant qc-type, hence it will work in the abstract case as well.

Assumption (*iii*) says that the abstract cup product coincides with the usual one on cohomology. By the above discussion reconstruction applies and we find that $G_{(\lambda)}$ is determined recursively by $\partial_{t_i}\partial_{t_j}G_{(\lambda)}|_{t_{\text{small}}}$, *with the same recursion coefficients* in $\mathbb{C}(\lambda)$ as those for $F^Y_{(\lambda)}$. By assumption (iv), we are done. □

We go back now to the proof of Theorem 4.1. We intend to apply Lemma 4.2.1 to $Y = Fl$, $G = \varphi^* F'$. Note that assumption (v) holds by Lemma 4.1.3, assumption (i) holds by (3.6.3), while (ii) and (iii) are immediate from the construction of F' in Sects. 3.2–3.5. Hence the theorem will be proved if we can show that (iv) holds as well, i.e.,

$$\text{(4.2.1)}\qquad \partial_{t_i}\partial_{t_j}F^{Fl,\mathbf{S}}\big|_{t_0=t_{r+1}=\cdots=t_{m-1}=0} = \varphi^*(\partial_{s_i}\partial_{s_j}F'|_{s_0=s_{r+1}=\cdots=s_{m-1}=0}).$$

Recall that (in the notation of Sect. 2.2)

$$\partial_{t_i}\partial_{t_j}F^{Fl,\mathbf{S}}(t) = \langle\langle\sigma_i,\sigma_j\rangle\rangle.$$

Setting $t_0 = t_{r+1} = \cdots = t_{m-1} = 0$ and using the divisor axiom we get

$$\text{(4.2.2)}\quad \partial_{t_i}\partial_{t_j}F^{Fl,\mathbf{S}}\big|_{t_0=t_{r+1}=\cdots=t_{m-1}=0} = \sum_{d=(d_1,\ldots,d_r)}\prod_{l=1}^{r}\left(Q_l e^{t_l}\right)^{d_l}\langle\sigma_i,\sigma_j\rangle^{Fl,\mathbf{S}}_{0,2,d}.$$

On the other hand, since

$$\partial_{s_k}\partial_{s_i}\partial_{s_j}F' = \partial_{s_k}((\partial_{\tilde{t}_i\cup\omega}\partial_{\tilde{t}_j\cup\omega}F)|_N)$$

it follows that

$$\text{(4.2.3)}\qquad \partial_{s_i}\partial_{s_j}F' = (\partial_{\tilde{t}_i\cup\omega}\partial_{\tilde{t}_j\cup\omega}F)|_N$$

up to a constant (in the base ring). By adding appropriate quadratic terms to F', we may assume that (4.2.3) holds exactly. (Recall that F is the $\mathbf{S}$-equivariant Gromov–Witten potential of $X//\mathbf{T}$ with the Novikov variables specialized as in (4.1.1).) Moreover, the first Chern class of the flag manifold satisfies

$$\int_d c_1(T(Fl)) = \sum_{l=1}^{r} d_l(k_{l+1}-k_{l-1}) \geq \min_l\{k_{l+1}-k_{l-1}\} \geq 2.$$

Therefore the specialization of the left-hand side of (4.2.3) to $s_0 = s_{r+1} = \cdots = s_{m-1} = 0$ is equal to its specialization at $\tilde{t}_0 = \tilde{t}_{r+1} = \cdots = \tilde{t}_{m-1} = 0$ by Lemma 3.6.1 (i). Using the divisor axiom as above in the right hand side of (4.2.3), we conclude that

$$\text{(4.2.4)}$$

$$\partial_{s_i}\partial_{s_j}F'\big|_{s_{r+1}=\cdots=s_{m-1}=0} = \sum_{d=(d_1,\ldots,d_r)}\prod_{l=1}^{r}\left(Q_l e^{\tilde{t}_l}\right)^{d_l}\Big(\sum_{\tilde{d}\mapsto d}(-1)^{\sum(k_l-1)d_l}\langle\gamma_i\cup\omega,\gamma_j\cup\omega\rangle^{X//\mathbf{T},\mathbf{S}}_{0,2,\tilde{d}}\Big).$$

Using (4.2.2) and (4.2.4), the proof of (4.2.1), and therefore of Theorem 4.1.1, is reduced to checking the following identity among 2-point invariants:

$$\langle \sigma_i, \sigma_j \rangle_{0,2,d}^{Fl,\mathbf{S}} = \sum_{\tilde{d} \mapsto d} (-1)^{\sum (k_l - 1) d_l} \langle \gamma_i \cup \omega, \gamma_j \cup \omega \rangle_{0,2,\tilde{d}}^{X//\mathbf{T},\mathbf{S}}. \tag{4.2.5}$$

4.3. Lee–Pandharipande reconstruction and reduction to 1-point invariants. There is another reconstruction theorem, due to Lee and Pandharipande [LP1], and independently to Bertram and Kley, which reduces in certain cases computations of (descendant) GW-invariants with any number of insertions to 1-point descendants. In fact, Lee and Pandharipande deduce the reconstruction result from universal relations they found among divisors in the Picard group of the moduli space $\overline{M}_{0,2}(\mathbb{P}^N, d[line])$ of 2-pointed stable maps to $\mathbb{P}^N$. We establish first a straightforward equivariant extension of their divisor relation.

Let Y be a projective variety with $\mathbf{S}$-action. Let $\overline{M}_{0,2}(Y, \beta)$ be the moduli space of 2-pointed genus zero stable maps with evaluation maps

$$ev_1, ev_2 : \overline{M}_{0,2}(Y, \beta) \longrightarrow Y.$$

The moduli space inherits an $\mathbf{S}$-action and the evaluation maps are equivariant. Let $\psi = \psi_1$ be the $\mathbf{S}$-equivariant first Chern class of the line bundle on $\overline{M}_{0,2}(Y, \beta)$ with fiber $T^*_{x_1}C$ over the point $[f : (C, x_1, x_2) \to Y]$.

There is a "boundary divisor" $D_{1,\beta_1|2,\beta_2}$ in $\overline{M}_{0,2}(Y, \beta)$ corresponding to maps with reducible domains and splitting type $\beta_1 + \beta_2 = \beta$. It is obtained as the image of the ($\mathbf{S}$-equivariant) gluing morphism

$$j_{\beta_1,\beta_2} : \overline{M}_{0,\{x_1,\bullet\}}(Y, \beta_1) \times_Y \overline{M}_{0,\{x_2,\bullet\}}(Y, \beta_2) \longrightarrow \overline{M}_{0,2}(Y, \beta)$$

and one defines its virtual fundamental class in the equivariant Chow group $A_*^{\mathbf{S}}(\overline{M}_{0,2}(Y, \beta), \mathbb{Q})$ as the push-forward of

$$[\overline{M}_{0,\{x_1,\bullet\}}(Y, \beta_1)]^{\mathrm{vir}} \boxtimes [\overline{M}_{0,\{x_2,\bullet\}}(Y, \beta_2)]^{\mathrm{vir}}.$$

Lemma 4.3.1. *For all $L \in \mathrm{Pic}^{\mathbf{S}}(Y)$, the relation*

$$ev_2^*(L) \cap [\overline{M}_{0,2}(Y, \beta)]^{\mathrm{vir}} - \Big(ev_1^*(L) + \Big(\int_\beta L\Big)\psi\Big) \cap [\overline{M}_{0,2}(Y, \beta)]^{\mathrm{vir}} + \sum_{\beta_1+\beta_2=\beta} \Big(\int_{\beta_2} L\Big) \cap [D_{1,\beta_1|2,\beta_2}]^{\mathrm{vir}} = 0$$

holds in $A_*^{\mathbf{S}}(\overline{M}_{0,2}(Y, \beta), \mathbb{Q})$.

Proof. As in [LP1], since the relation is linear in L and the equivariant Picard group is spanned over $\mathbb{Q}$ by $\mathbf{S}$-equivariant very ample line bundles, the Lemma will follow from the case $Y = \mathbb{P}^N$, $\beta = d[line]$ and the stronger statement

$$(4.3.1) \quad ev_2^*(L) - ev_1^*(L) - \Big(\int_\beta L\Big)\psi + \sum_{\beta_1+\beta_2=\beta} \Big(\int_{\beta_2} L\Big) D_{1,\beta_1|2,\beta_2} = 0$$

in $\mathrm{Pic}^{\mathbf{S}}(\mathbb{P}^N)$.

The relation (4.3.1) holds after passing to the non-equivariant limit $\lambda_i = 0$ by [LP1, Theorem 1]. Therefore the left-hand side is a linear polynomial in the λ_i's and the corresponding equivariant line bundle is just a trivial bundle twisted by a character of $\mathbf{S}$. To check that this character is trivial, it suffices to restrict to *any* $\mathbf{S}$-fixed point of $\overline{M}_{0,2}(\mathbb{P}^N, d[line])$. There are many possible choices of fixed points that will work. One particular such for which the computation is very easy is the point corresponding to a stable map with domain $C \cup D$ (the union of two irreducible components) such that $x_1, x_2 \in C$ and $f : C \cup D \to \mathbb{P}^N$ collapses C to a fixed point $p \in \mathbb{P}^N$ and maps D with degree d onto an $\mathbf{S}$-invariant line in $\mathbb{P}^N$ joining p to another fixed point q, such that the map is totally ramified at q. The classes ψ and $D_{1,\beta_1|2,\beta_2}$ vanish when restricted to this point, while $ev_1^* L$ and $ev_2^* L$ have the same restriction. Relation (4.3.1) and hence the lemma are proved. □

We will use Lemma 4.3.1 to obtain a reconstruction result in the context of the abelian/nonabelian correspondence. Recall that descendant (genus 0) Gromov–Witten invariants of a smooth projective Y are defined by

$$\langle \tau_{a_1}(\gamma_1), \ldots, \tau_{a_n}(\gamma_n)\rangle^Y_{0,n,\beta} := \int_{[M_{0,n}(Y,\beta)]^{\mathrm{vir}}} \prod_i \psi_i^{a_i} ev_i^*(\gamma_i),$$

where $\gamma_i \in H^*(Y)$ and ψ_i are the first Chern classes of the cotangent line bundles at the marked points. The definition extends to torus-equivariant descendants (which will be $\mathbb{C}[\lambda]$-valued). We establish first an auxiliary vanishing result for certain descendant invariants of $X//\mathbf{T}$.

Let $X//\mathbf{G}$, $X//\mathbf{T}$, $\mathbf{S}$ etc. be as in the setting Sect. 3.1. Let $\beta \in H_2(X//\mathbf{G}, \mathbb{Z})$ be fixed. Consider the moduli space

$$\mathcal{M}_\beta := \coprod_{\tilde\beta \mapsto \beta} \overline{M}_{0,n}(X//\mathbf{T}, \tilde\beta)$$

with the obvious evaluation maps $ev_i : \mathcal{M}_\beta \longrightarrow X//\mathbf{T}$, $i = 1, \ldots, n$ and virtual class $[\mathcal{M}_\beta]^{\mathrm{vir}}$. Note that

$$H^*_{\mathbf{S}}(\mathcal{M}_\beta, \mathbb{C}) \cong \bigoplus_{\tilde\beta \mapsto \beta} H^*_{\mathbf{S}}(\overline{M}_{0,n}(X//\mathbf{T}, \tilde\beta), \mathbb{C}).$$

Introduce "psi-classes" on $\mathcal{M}_\beta$ by

$$\psi_i := \sum_{\tilde{\beta}\mapsto\beta} \psi_{i,\tilde{\beta}}$$

and define for cohomology classes $\gamma_1, \dots, \gamma_n \in H^*_{\mathbf{S}}(X//\mathbf{T})$.

$$\begin{aligned} I_{n,\beta}(\tau_{a_1}(\gamma_1), \dots, \tau_{a_n}(\gamma_n)) &:= (-1)^{\epsilon(\beta)} \int_{[\mathcal{M}_\beta]^{\mathrm{vir}}} \prod_i \psi_i^{a_i} ev_i^*(\gamma_i) \\ &= (-1)^{\epsilon(\beta)} \sum_{\tilde{\beta}\mapsto\beta} \langle \tau_{a_1}(\gamma_1), \dots, \tau_{a_n}(\gamma_n)\rangle_{0,n,\tilde{\beta}}^{X//\mathbf{T},\mathbf{S}}. \end{aligned}$$

Recall that the intersection form is non-degenerate on the $\mathbf{W}$-anti-invariant subspace $H^*_{\mathbf{S}}(X//\mathbf{T})^a$. We denote the orthogonal complement by $(H^*_{\mathbf{S}}(X//\mathbf{T})^a)^\perp$.

Lemma 4.3.2. *If $\tilde{\sigma}_1, \dots, \tilde{\sigma}_{n-1}$ are $\mathbf{W}$-invariant lifts of classes σ_i in $H^*_{\mathbf{S}}(X//\mathbf{G})$ and $\gamma \in (H^*_{\mathbf{S}}(X//\mathbf{T})^a)^\perp$, then*

$$I_{n,\beta}(\tau_{a_1}(\tilde{\sigma}_1 \cup \omega), \tau_{a_2}(\tilde{\sigma}_2), \dots, \tau_{a_{n-1}}(\tilde{\sigma}_{n-1}), \tau_{a_n}(\gamma)) = 0.$$

Proof. The $\mathbf{W}$-action on $X//\mathbf{T}$ induces a $\mathbf{W}$-action on $\mathcal{M}_\beta$, by composing stable maps with the automorphisms in $\mathbf{W}$. The evaluation maps are easily seen to be $\mathbf{W}$-equivariant. Note also that the psi-classes are $\mathbf{W}$-invariant. Hence the class

$$(ev_n)_*\Big(ev_1^*(\omega) \prod_{i=1}^{n} \psi_i^{a_i} \prod_{i=1}^{n-1} ev_i^*(\tilde{\sigma}_i) \cap [\mathcal{M}_\beta]^{\mathrm{vir}}\Big)$$

is $\mathbf{W}$-anti-invariant. The lemma now follows from the projection formula. □

Proposition 4.3.3. *Let $X//\mathbf{G}$, $X//\mathbf{T}$, $\mathbf{S}$ be as in the setting Sect. 3.1. Assume that the localized equivariant cohomology $H^*_{\mathbf{S}}(X//\mathbf{G}, \mathbb{C}) \otimes_{\mathbb{C}[\lambda]} \mathbb{C}(\lambda)$ is generated as a $\mathbb{C}(\lambda)$-algebra by divisors (that is, by $c_1(L) \otimes 1$ for $L \in \mathrm{Pic}^{\mathbf{S}}(X//\mathbf{G})$). Let σ_i, σ_j be any equivariant cohomology classes on $X//\mathbf{G}$, with $\mathbf{W}$-invariant lifts $\tilde{\sigma}_i, \tilde{\sigma}_j$ to $X//\mathbf{T}$. If the identity*

$$\langle \tau_a(\sigma_i), \sigma_j\rangle_{0,2,\beta}^{X//\mathbf{G},\mathbf{S}} = \sum_{\tilde{\beta}\mapsto\beta} (-1)^{\epsilon(\beta)} \langle \tau_a(\tilde{\sigma}_i \cup \omega), \tilde{\sigma}_j \cup \omega\rangle_{0,2,\tilde{\beta}}^{X//\mathbf{T},\mathbf{S}} \tag{4.3.2}$$

holds for $\sigma_j = 1$, then it holds in general.

Proof. It is enough to prove the Proposition for a *fixed* choice of lifts of cohomology classes on $X//\mathbf{G}$ to $X//\mathbf{T}$.

It follows immediately from Martin's integration formula (3.1.3) that

$$\widetilde{\sigma' \cup \sigma''} \cup \omega = \tilde{\sigma}' \cup \tilde{\sigma}'' \cup \omega \tag{4.3.3}$$

for any $\sigma', \sigma'' \in H^*_{\mathbf{S}}(X//\mathbf{G}, \mathbb{C})$ (see e.g. [BCK1, Cor. 2.3] for an argument).

Assume first that the equivariant cohomology ring of $X//\mathbf{G}$ is generated by divisors without localization (this happens for example when $X//\mathbf{G}$ is the complete flag manifold $Fl(1, 2, \ldots, n-1, n)$). Using Lemma 4.3.1 and the splitting axiom for GW-invariants we find that $\langle \tau_a(\sigma_i), \sigma_j \rangle_{0,2,\beta}^{X//\mathbf{G},\mathbf{S}}$ is expressed recursively (with $\mathbb{C}[\lambda]$-coefficients) in terms of invariants $\langle \tau_{a'}(\sigma'), 1 \rangle_{0,2,\beta'}^{X//\mathbf{G},\mathbf{S}}$ (these can be further reduced to 1-point descendants by the fundamental class axiom for GW-invariants). This is just the reconstruction of Lee-Pandharipande.

Recall the notation $I_{2,\beta}(\tau_a(\tilde{\sigma}_i \cup \omega), \tilde{\sigma}_j \cup \omega)$ introduced above for the right-hand side of the identity (4.3.2). The divisor relation in Lemma 4.3.1 can be extended in an obvious manner to the moduli space $\mathcal{M}_\beta$ for $\mathbf{W}$-invariant lifts $\tilde{L}$ of line bundles $L \in \mathrm{Pic}^{\mathbf{S}}(X//\mathbf{G})$. By Lemma 4.3.2, the reconstruction procedure applies to the invariants $I_{2,\beta}(\tau_a(\tilde{\sigma}_i \cup \omega), \tilde{\sigma}_j \cup \omega)$ and (by (4.3.3) and the equality $\epsilon(\beta_1 + \beta_2) = \epsilon(\beta_1) + \epsilon(\beta_2)$) it expresses them in terms of $I_{2,\beta'}(\tau_{a'}(\tilde{\sigma}' \cup \omega), \omega)$ with the <u>same</u> recursion coefficients. The proposition is proved in this case.

In the general case the same argument will work word for word, except that the recursion coefficients will now be rational functions rather than polynomials in the λ_i's. □

Remark 4.3.4. In view of Lemma 4.3.2, one might be tempted to try to extend the version of Lee–Pandharipande reconstruction above to descendants with any number of insertions and ψ-classes at all points. However, this is not possible, because an analogue of the fundamental class axiom does not hold for the invariants $I_{n,\beta}$ (indeed, in general $I_{n,\beta}(\tilde{\sigma}_1 \cup \omega, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_{n-1}, \omega) \neq 0$). This is the reason for which the "twisting" by $\star\, \omega$ is necessary.

Corollary 4.3.5. *The following identity holds between Gromov–Witten invariants of $X//\mathbf{G} = Fl(k_1, \ldots, k_r, n)$ and those of the abelian quotient $X//\mathbf{T}$: for any $d = (d_1, \ldots, d_r) \in H_2(Fl, \mathbb{Z})$, any $a \geq 0$ and any equivariant cohomology classes σ_i, σ_j on Fl, with lifts γ_i, γ_j respectively,*

$$\langle \tau_a(\sigma_i), \sigma_j \rangle_{0,2,d}^{Fl,\mathbf{S}} = \sum_{\tilde{d} \mapsto d} (-1)^{\sum (k_a - 1) d_a} \langle \tau_a(\gamma_i \cup \omega), \gamma_j \cup \omega \rangle_{0,2,\tilde{d}}^{X//\mathbf{T},\mathbf{S}}.$$

Proof. By Lemmas 4.1.3 and 4.3.1, it suffices to check that

$$\langle \tau_a(\sigma_i), 1 \rangle_{0,2,d}^{Fl,\mathbf{S}} = \sum_{\tilde{d} \mapsto d} (-1)^{\sum (k_a - 1) d_a} \langle \tau_a(\gamma_i \cup \omega), \omega \rangle_{0,2,\tilde{d}}^{X//\mathbf{T},\mathbf{S}}. \tag{4.3.4}$$

This is (essentially) proved in [BCK1,BCK2]. However, since the actual statement is explicitly written (see formula (5) on p. 124 and Remark on p. 125 in [BCK1]) only for Grassmannians and non-equivariant invariants, we should say a few words here.

For the general flag manifold, a correspondence between the equivariant "small" J-functions of Fl and $X//\mathbf{T}$ is given by [BCK2, Theorem 1] (see the next section below for more about J-functions). Reading the argument in [BCK1, pp. 124–125] backwards[1], the equality (4.3.4) follows from the J-functions correspondence, provided that for any factorization

$$\omega = \left(\sqrt{\frac{(-1)^{|\Phi_+|}}{|\mathbf{W}|}}\right)\Big(\prod_{\alpha\in A} c_1^{\mathbf{S}}(L_\alpha)\Big) \cup \Big(\prod_{\alpha\in\Phi_+\setminus A} c_1^{\mathbf{S}}(L_\alpha)\Big)$$

we have

$$\left(\sqrt{\frac{(-1)^{|\Phi_+|}}{|\mathbf{W}|}}\right)\Big(\prod_{\alpha\in A} c_1^{\mathbf{S}}(L_\alpha)\Big) \star_{\mathrm{small}} \Big(\prod_{\alpha\in\Phi_+\setminus A} c_1^{\mathbf{S}}(L_\alpha)\Big) = \omega, \tag{4.3.5}$$

where $\star_{\mathrm{small}}$ is the small equivariant quantum product on $X//\mathbf{T}$, restricted to $H^2_{\mathbf{S}}(X//\mathbf{T}, \mathbb{C})^{\mathbf{W}}$ and with the Novikov variables specialized as in (4.1.1). By a simple degree counting, this last equality is always true when $X//\mathbf{G}$ (and hence $X//\mathbf{T}$, cf. Sect. 3.4) is a Fano variety. Indeed, the left-hand side of (4.3.5) is $\mathbf{W}$-anti-invariant, homogeneous, and of the form

$$\omega + \text{quantum corrections}.$$

However, ω is the unique class of lowest degree in $H^*_{\mathbf{S}}(X//\mathbf{T}, \mathbb{C})^a$, and in the Fano case the quantum parameters have positive degree. Hence the quantum corrections must vanish. □

It remains to observe that relation (4.2.5) is a special case of the corollary to conclude the proof of Theorem 4.1.1. □

Note that the only instance in this section where we have used that $X//\mathbf{G}$ is a flag manifold was in quoting the small J-function correspondence from [BCK2]. In other words, we have proved

Theorem 4.3.6. *Let X, $\mathbf{G}$, $\mathbf{T}$, $\mathbf{S}$ etc. be as in the setting Sect. 3.1. Assume that $X//\mathbf{G}$ is Fano of index ≥ 2 and that its equivariant cohomology is generated by divisors after localization. Then Conjecture 3.7.1 holds if and only if* (4.3.4) *holds, if and only if the abelian/nonabelian correspondence for small J-functions holds.*

---

[1] The specialization of the t_i-variables there corresponds exactly to our specialization (4.1.1) of the Novikov variables Q_i here.

A similar statement holds if we only assume that $c_1(T(X//\mathbf{G}))$ is nef, by using Lemma 3.6.1 *(ii)* in the argument just above (4.2.4). However, the change of coordinates $s(\tilde{t})$ will be nontrivial even for the restriction to subspace $\{s_{r+1} = \cdots = s_{m-1} = 0\}$, and coincides with the change of coordinates in the abelian/nonabelian correspondence for small J-functions (see [BCK2, Conjecture 4.3]). This is precisely analogous to the mirror theorem [Giv1] for hypersurfaces in projective space. We leave the precise formulation for the interested reader.

5. The abelian/nonabelian correspondence for J-functions

Our goal in this section is to explain why Conjecture 3.7.1 is equivalent to (an extension to the big parameter space of) the correspondence between the J-functions of $X//\mathbf{G}$ and $X//\mathbf{T}$ proposed in [BCK2, Conjecture 4.3]. In particular, by Theorem 4.1.1 and Corollary 5.3.4 below, the correspondence holds for the flag manifolds $Fl(k_1, \ldots, k_r, n)$.

5.1. Deformed flat coordinates. First we recall the definition of deformed flat coordinates following Dubrovin [Du1,Du2,Du3]. Let M be a Frobenius manifold (say, analytic, for simplicity), with Euler vector field. There is a deformed flat connection ∇^z on Θ_M given by

$$\nabla^z_\xi \eta := \nabla_\xi \eta - z^{-1} \xi \star \eta$$

(see [Du1, p. 189 and p. 323] and also [Du3]; however, we follow Givental for the convention on z). By identifying the cotangent sheaf Ω^1_M and the tangent sheaf Θ_M via the flat metric, a deformed flat connection is induced on Ω^1_M. A coordinate system J_i of M is called a *deformed flat coordinate system* if dJ_i are horizontal sections. In other words, J_i form a complete solution space to the second order linear PDE system

$$z \partial_{t_i} \partial_{t_j} J = \sum_{\gamma} c_{ij}^k \partial_{t_k} J \tag{5.1.1}$$

where t_i are flat coordinates and c_{ij}^k are structure constants of multiplications, i.e., $\partial_{t_i} \star \partial_{t_j} = \sum_k c_{ij}^k \partial_{t_k}$.

Suppose that the potential function F (defined up to quadratic terms) for the Frobenius structure is of the form $F = F_c + F_q$, with F_c a cubic form of the flat coordinates t_i and $F_q \in \mathbb{C}[[q_1, \ldots, q_r, t_{r+1}, \ldots, t_R]]$ such that $q_i = e^{t_i}$ and $F_q \equiv 0$ modulo the ideal $(q_1, \ldots, q_r)$ (cf. 2.1.1).

Consider the normalization condition

$$\sum J_i \partial_{t_i} \equiv z e^{t/z} = z\partial_{t_0} + t + O(z^{-1}) \quad (\text{mod } (q_1, \ldots, q_r)),$$

where $t = \sum t_i \partial_{t_i}$, the products of vector fields in the exponential are the "cup" products (determined by $F_{cl} = (1/6) g(t \cup t, t)$) and $1 = \partial_{t_0}$. The normalization uniquely determines deformed flat coordinates once the flat coordinates are chosen (see [Du2, Lemma 2.2]). We will call $\sum J_i \partial_i$ the *J-function* if it is normalized as above.

5.2. J-functions in quantum cohomology. The J-function plays an important role in Gromov–Witten theory. Let Y be a projective algebraic manifold. Then the J-function J_Y for the (formal) Frobenius structure defined by the quantum cohomology of Y can be constructed using descendant Gromov–Witten invariants. Let $\{\phi_i\}$ be a homogeneous basis of $H^*(Y)$, with Poincaré dual basis $\{\phi^i\}$. Let $t := \sum_i t_i \phi_i$. J_Y coincides with the assignment

$$(5.2.1) \qquad H^*(Y) \ni t \mapsto z + t + \sum_i \phi^i \left\langle\!\!\left\langle \frac{\phi_i}{z-\psi} \right\rangle\!\!\right\rangle \quad \in z + t + \mathcal{H}_-$$

cf. [CG,Giv3], where $\mathcal{H}_- = \frac{1}{z} H^*(Y) \otimes_{\mathbb{C}} N[Y][[\frac{1}{z}]]$.

Here we use the double-bracket notation introduced in Sect. 2.2, so that

$$\left\langle\!\!\left\langle \frac{\phi_i}{z-\psi} \right\rangle\!\!\right\rangle = \sum_{\beta \in NE_1} Q^\beta \sum_{n \geq 0} \frac{1}{n!} \int_{[\overline{M}_{0,n+1}(Y,\beta)]^{\text{vir}}} \frac{ev_1^*(\phi_i)}{z-\psi} ev_2^*(t) \ldots ev_{n+1}^*(t)$$

where $\psi = \psi_1$ and $1/(z-\psi)$ is formally expanded as a geometric series.

The normalization condition

$$J_Y(t,z) \equiv z e^{t/z}$$

modulo quantum corrections follows from the well-known result

$$\int_{\overline{M}_{0,n}} \psi_1^{l_1} \ldots \psi_n^{l_n} = (n-3)!/l_1! \ldots l_n! \quad \text{if } \sum l_i = n-3.$$

(Note that in the paper [BCK2] $J_Y(t,z)/z$ is used for J-function, i.e., a different normalization.)

5.3. The abelian/nonabelian correspondence. Let X, $\mathbf{G}$, $\mathbf{T}$ be as in the setting Sect. 3.1. (For simplicity, we do not consider the equivariant theory here; the interested reader can readily make the necessary modifications to cover this case as well.) We have the $\mathbf{W}$-induced Frobenius structure over the Novikov ring $N(X//\mathbf{G})$ constructed in Sects. 3.2–3.6. We will keep the notations, and make liberal use of all its properties proved there. Moreover, *from now on, we assume that Conjecture 3.7.1 holds for $X//\mathbf{G}$ and $X//\mathbf{T}$.*

If $J_{X//\mathbf{G}} = \sum_{i=0}^{m-1} J_{i,X//\mathbf{G}}(t_0, \ldots, t_{m-1}, z)\sigma_i$ is the J function of $X//\mathbf{G}$, as given by (5.2.1), put

$$\tilde{J}_{X//\mathbf{G}}(t,z) := \sum_{i=0}^{m-1} J_{i,X//\mathbf{G}}(t_0, \ldots, t_{m-1}, z)\gamma_i.$$

(Recall that γ_i's are chosen $\mathbf{W}$-invariant lifts of the σ_i's.)

Lemma 5.3.1. $\tilde{J}_{X//G}(t,z) \cup \omega = (z\partial_\omega J_{X//T})\big|_{Q^{\tilde{\beta}} = (-1)^{\epsilon(\beta)}Q^{\beta}, N}(\varphi(t), z)$.

Proof. Both sides satisfy the normalization condition $J \equiv ze^{t/z} \cup \omega$ modulo quantum corrections. Therefore it suffices to check that $\{\partial_\omega J_i\}_i$ forms a deformed flat coordinate system for $(N, \circ, {}^{\omega}g, e, \mathfrak{E})$ if J_δ is a deformed flat coordinate for $(M, \star, g, e, \mathfrak{E})$ such that $\{J_i|_N\}_i$ form a coordinate system of N. Indeed, by Conjecture 3.7.1, which we're assuming, the Frobenius manifolds P and N are isomorphic via φ.

First, we rewrite the PDE (5.1.1) as

$$z\partial_i\partial_j J = (\partial_i \star \partial_j)J. \tag{5.3.1}$$

This is useful in computations.

Next, if ξ and η are ${}^{\omega}\nabla$-horizontal vector fields, then

$$\begin{aligned} z\partial_{\xi\circ\eta}\partial_\omega J_i &= \partial_{(\xi\circ\eta)\star\omega} J_i \\ &= \partial_{\xi\star(\eta\star\omega)} J_i \\ &= z\partial_\xi \partial_{\eta\star\omega} J_i \\ &= z^2 \partial_\xi \partial_\eta \partial_\omega J_i \end{aligned}$$

since ω and $\eta \star \omega$ are ∇-horizontal. □

Remark 5.3.2. Lemma 5.3.1 reveals the relation between $\tilde{t}$ and the $s = \varphi(t)$:

$$\tilde{t} = s + \sum_{n=0}^{\infty} \frac{(-1)^{\epsilon(\beta)}Q^\beta}{n!} \sum_{i,\ 0\neq\tilde{\beta}\mapsto\beta} \gamma_i < \gamma^i \cup \omega, \omega, \underbrace{s, \ldots, s}_{n} >_{0,n+2,\tilde{\beta}},$$

where $\{\gamma^j \cup \omega\}$ is the basis of $H^*(X//\mathbf{T})^a$ dual to $\{\gamma_i \cup \omega\}$, that is,

$$\int_{X//\mathbf{T}} \gamma_i \cup \omega \cup \gamma^j \cup \omega = \delta_i^j.$$

Define, for $\tau \in N$,

(5.3.2)

$$\begin{aligned} I(\tau, z) :&= \Big(\Big(\prod_{\alpha\in\Phi_+} z\partial_\alpha\Big) J_{X//T}\Big)\Big|_{Q^{\tilde{\beta}} = (-1)^{\epsilon(\beta)}Q^\beta, N}(\tau, z) \\ &= \sum_\beta (-1)^{\epsilon(\beta)} Q^\beta \sum_{\tilde{\beta}\mapsto\beta} \prod_{\alpha\in\Phi_+} \Big(c_1(L_\alpha) + z\int_{\tilde{\beta}} c_1(L_\alpha)\Big) J^{\tilde{\beta}}_{X//\mathbf{T}}\Big|_N(\tau, z) \end{aligned}$$

where ∂_α is the (∇-flat) vector field associated to $c_1(L_\alpha)$, the derivative of J is taken component-wise and $J^{\tilde{\beta}}_{X//\mathbf{T}}$ is the coefficient of $Q^{\tilde{\beta}}$ in $J_{X//\mathbf{T}}$ before specializing the Novikov variables. The latter equality follows from the divisor axiom.

Theorem 5.3.3. *There are unique* $C^i(t,z) \in N(X//\mathbf{G})[z][[t]]$ *such that*

$$I(\varphi(t),z) = \sum_i C^i(t,z) z\partial_{t_i} \tilde{J}_{X//\mathbf{G}}(t,z) \cup \omega.$$

Proof. For the proof we use Givental's description [Giv3] of the rational Gromov–Witten theory for a projective manifold Y by means of a certain Lagrangian cone $\mathcal{L}_Y$ with special properties (see [Giv3, Theorem 1]).

Let $s \in N$. By the very definition

$$I(s,-z) := \pm\Big(\Big(\prod_{\alpha\in\Phi_+} z\partial_\alpha\Big) J_{X//\mathbf{T}}\Big)\Big|_{Q^{\tilde\beta}=(-1)^{\epsilon(\beta)}Q^\beta}(s,-z) \in zL,$$

where $zL := zT_p\mathcal{L}_{X//\mathbf{T}}$ is the tangent space to the Lagrangian cone at the point $p = J_{X//\mathbf{T}}(s)$.

Let $\{\phi_\mu\}$ be a basis of $H^*(X//\mathbf{T})$ obtained by adjoining to the basis $\{\gamma_i \cup \omega\}$ of the $\mathbf{W}$-anti-invariant subspace $H^*(X//\mathbf{T})^a$ a basis of $(H^*(X//\mathbf{T})^a)^\perp$. Since $\{z\partial_\mu J_{X//\mathbf{T}}(s,-z)\}$ form a basis of zL/z^2L over $N(X//\mathbf{T})$,

$$I(s,z) = \sum C^\mu(s,z) z\partial_\mu J_{X//\mathbf{T}}|_{Q^{\tilde\beta}=(-1)^{\epsilon(\beta)}Q^\beta}(s,z)$$

for some unique $C^\mu(s,z) \in N(X//\mathbf{T})[z][[s]]$.

Since I is $\mathbf{W}$-anti-invariant by construction, the terms corresponding to the basis of $(H^*(X//\mathbf{T})^a)^\perp$ must vanish and we obtain

$$I(s,z) = \sum_i C^{\gamma_i\cup\omega}(s,z)(z\partial_{\gamma_i\cup\omega} J_{X//\mathbf{T}})|_{Q^{\tilde\beta}=(-1)^{\epsilon(\beta)}Q^\beta}(s,z). \tag{5.3.3}$$

Now $\partial_{s_i} \star \omega = \partial_{\gamma_i\cup\omega}$, therefore by (5.3.1)

$$\begin{aligned}\sum_i C^{\gamma_i\cup\omega}(s,z)&(z\partial_{\gamma_i\cup\omega} J_{X//\mathbf{T}})|_{Q^{\tilde\beta}=(-1)^{\epsilon(\beta)}Q^\beta}(s,z)\\ &= \sum_i C^{\gamma_i\cup\omega}(s,z)(z\partial_{s_i} z\partial_\omega J_{X//\mathbf{T}})|_{Q^{\tilde\beta}=(-1)^{\epsilon(\beta)}Q^\beta}(s,z).\end{aligned} \tag{5.3.4}$$

Finally, Lemma 5.3.1 gives

$$\begin{aligned}\sum_i C^{\gamma_i\cup\omega}(s,z)&(z\partial_{s_i} z\partial_\omega J_{X//\mathbf{T}})|_{Q^{\tilde\beta}=(-1)^{\epsilon(\beta)}Q^\beta}(s,z)\\ &= \sum_i C^{\gamma_i\cup\omega}(t,z) z\partial_{t_i} \tilde{J}_{X//\mathbf{G}}(t,z) \cup \omega,\end{aligned} \tag{5.3.5}$$

where $\varphi(t) = s$. The theorem follows from (5.3.3), (5.3.4) and (5.3.5). $\square$

Corollary 5.3.4.

$$\tilde{J}_{X//\mathbf{G}}(t,z)\cup\omega = I(\tilde{t},z)+\sum_i C^i(\tilde{t},z)z\partial_{\tilde{t}_i}I(\tilde{t},z)$$

for some unique $C^i(\tilde{t},z)\in N(X//G)[[z,\tilde{t}]]$, *where* $\tilde{t}=\sum\tilde{t}_i\gamma_i$. *The expression of* t *in terms of* $\tilde{t}$ *is uniquely determined by the expansion of the right-hand side as* $z+t(\tilde{t})+O(z^{-1})$ *(and coincides with the formula* (3.6.2)*).*

Proof. The theorem above shows that, with the identification of cohomology spaces $H^*(X//\mathbf{T})^a$ with $H^*(X//\mathbf{G})$ by the map $\tilde{\sigma}\cup\omega\mapsto\sigma$, the I-function generates the Lagrangian cone $\mathcal{L}_{X//\mathbf{G}}$ describing the rational Gromov–Witten theory of $X//\mathbf{G}$ [Giv3]. Since $\{z\partial_{\tilde{t}_i}I(\tilde{t},-z)\}$ also form a basis of L/zL, where L is the tangent space of $\mathcal{L}_{X//\mathbf{G}}$ at the point $I(\tilde{t},-z)$, the corollary follows.

A constructive argument may also be given, using the "Birkhoff factorization" method. See [CG], Corollary 5 and the paragraph before it for details. □

Corollary 5.3.4 is a generalization of [BCK2, Conjecture 4.3] to the "big" parameter space. The arguments in this section can be reversed to show that the corollary implies Conjecture 3.7.1

6. Flag manifolds for other classical types

In this section, we extend the abelian/nonabelian correspondence in the presence of additional twists by homogeneous vector bundles and apply it to the case of generalized flag manifolds of Lie groups of types B, C, D.

6.1. Twisting by bundles. Let $\mathbf{S}\times\mathbf{G}$ act on X as in Sect. 3.1. Let $\mathcal{V}$ be a $\mathbf{G}$-representation space (as in [BCK2]). There are $\mathbf{S}\times\mathbf{G}\times\mathbb{C}^*$-actions on X and $\mathcal{V}$ (where $\mathbb{C}^*$ acts trivially on X and homothetically on $\mathcal{V}$), inducing $\mathbf{S}\times\mathbb{C}^*$-equivariant vector bundles

$$\mathcal{V}_{\mathbf{T}}:=X^s(\mathbf{T})\times_{\mathbf{T}}\mathcal{V},\quad \mathcal{V}_{\mathbf{G}}:=X^s(\mathbf{G})\times_{\mathbf{G}}\mathcal{V}$$

over nonsingular quotients $X//\mathbf{T}$ and $X//\mathbf{G}$, respectively. Put $\mathbb{C}[\lambda']:=H^*(B\mathbb{C}^*)$.

There is an $\mathbf{S}\times\mathbb{C}^*$-equivariant Frobenius structure on

$$Z':=Spf\Bigg(N(X//\mathbf{T})[\lambda]\otimes_{\mathbb{C}}\mathbb{C}\left(\left(\frac{1}{\lambda'}\right)\right)\left[\left[\left(H^*_{\mathbf{S}}(X//\mathbf{T})\otimes\left(N(X//\mathbf{T})[\lambda]\otimes_{\mathbb{C}}\mathbb{C}\left(\left(\frac{1}{\lambda'}\right)\right)\right)\right)^\vee\right]\right]\Bigg)$$

defined by the $\mathbf{S}\times\mathbb{C}^*$-equivariant genus zero Gromov–Witten invariants of $X//\mathbf{T}$ *twisted by (the equivariant Euler class of)* $\mathcal{V}_\mathbf{T}$. Here we introduce the extra coefficient ring $\mathbb{C}((\frac{1}{\lambda'}))$ to invert

$$c_{\mathrm{top}}^{\mathbf{S}\times\mathbb{C}^*}(\mathcal{V}_\mathbf{T})=\sum_{i=0}^{\mathrm{rk}\mathcal{V}_\mathbf{T}}(\lambda')^{\mathrm{rk}\mathcal{V}_\mathbf{T}-i}c_i^S(\mathcal{V}_\mathbf{T}).$$

We list some comments on this Frobenius structure for clarification, and refer the reader to [CG] for details.

- The twisted metric $g_{\mathcal{V}_\mathbf{T}}$ is given by

$$g_{\mathcal{V}_\mathbf{T}}(a,b):=\int_{X//\mathbf{T}}a\cup b\cup c_{\mathrm{top}}^{\mathbf{S}\times\mathbb{C}^*}(\mathcal{V}_\mathbf{T}),\quad\text{for } a,b\in H_\mathbf{S}^*(X//\mathbf{T}).$$

- The twisted product is given by the requirement that

$$\begin{aligned}&g_{\mathcal{V}_\mathbf{T}}(a*_{\mathcal{V}_\mathbf{T}}b,\ c)=\langle\langle a,b,c\rangle\rangle_{\mathcal{V}_\mathbf{T}}\\&:=\sum_{\tilde\beta\in NE_1(X//\mathbf{T})}\sum_n\frac{Q^{\tilde\beta}}{n!}\int_{[\overline{M}_{0,n+3}(X//\mathbf{T},\tilde\beta)]^{\mathrm{vir}}}ev_1^*(a)ev_2^*(b)ev_3^*(c)\\&\qquad\times ev_4^*(t)\ \ldots\ ev_{n+3}^*(t)\ c_{\mathrm{vir.top}}^{\mathbf{S}\times\mathbb{C}^*}(R^\bullet\pi_*ev_{n+4}^*\mathcal{V}_\mathbf{T}),\end{aligned}$$

  where π denotes the projection $\overline{M}_{0,n+4}(X//\mathbf{T},\tilde\beta)\to\overline{M}_{0,n+3}(X//\mathbf{T},\tilde\beta)$ of moduli stacks of stable maps which forgets the last marked point.
- The Euler vector field is $\mathfrak{E}_{\mathcal{V}_\mathbf{T}}=\mathfrak{E}+\mathcal{E}_\mathbf{S}+\mathcal{E}_{\mathbb{C}^*}-c_1^\mathbf{S}(\mathcal{V}_\mathbf{T})$.
- The normalized ($\mathbf{S}\times\mathbb{C}^*$-equivariant) J-function is

$$J_{\mathcal{V}_\mathbf{T}}^{\mathbf{S}\times\mathbb{C}^*}:t\mapsto z+t+\sum_i\phi^i\left\langle\left\langle\frac{\phi_i}{z-\psi}\right\rangle\right\rangle_{\mathcal{V}_\mathbf{T}},$$

  where $\{\phi_i\}$ and $\{\phi^i\}$ are dual bases with respect to the twisted metric $g_{\mathcal{V}_\mathbf{T}}$.

Similarly, we construct an $\mathbf{S}\times\mathbb{C}^*$-equivariant Frobenius structure on the formal scheme P' associated to $H_\mathbf{S}^*(X//\mathbf{G})\otimes(N(X//\mathbf{G})[\lambda]\otimes\mathbb{C}((\frac{1}{\lambda'})))$ using genus zero $\mathbf{S}\times\mathbb{C}^*$-equivariant Gromov–Witten invariants on $X//\mathbf{G}$ twisted by $\mathcal{V}_\mathbf{G}$.

Now, as in Sect. 3.2, we can further twist the Frobenius structure on Z' by $\omega:=\sqrt{\frac{1}{|\mathbf{W}|}\prod_{\alpha\in\Phi}c_1^\mathbf{S}(L_\alpha)}$ in order to induce an $\mathbf{S}\times\mathbb{C}^*$-equivariant Frobenius structure on the formal scheme N' over $N(X//\mathbf{G})[\lambda]\otimes_\mathbb{C}\mathbb{C}((\frac{1}{\lambda'}))$ obtained as in loc. cit. by fixing a lift of $H_\mathbf{S}^*(X//\mathbf{G})$ to $H_\mathbf{S}^*(X//\mathbf{T})^\mathbf{W}$.

Conjecture 6.1.1. Let $\varphi:P'\to N'$ be the isomorphisms of formal schemes over $N(X//G)[\lambda]\otimes_\mathbb{C}\mathbb{C}((\frac{1}{\lambda'}))$ defined by $\varphi^*(s_i)=t_i$. Then φ induces an isomorphism of formal $\mathbf{S}\times\mathbb{C}^*$-equivariant Frobenius structures.

Theorem 6.1.2. *Conjecture 3.7.1 implies Conjecture 6.1.1, and furthermore,*

$$\tilde{J}^{\mathbf{S}\times\mathbb{C}^*}_{\mathcal{V}_\mathbf{G}}(t,z)\cup\omega = z\partial_\omega J^{\mathbf{S}\times\mathbb{C}^*}_{\mathcal{V}_\mathbf{T}}\big|_{Q^{\tilde\beta}=(-1)^{\epsilon(\beta)}Q^\beta,N'}(\varphi(t),z).$$

Proof. It is enough to show the equality of J-functions above, since it implies that φ preserves the product structures.

Abusing notation, for $\gamma\in H^*_{\mathbf{S}\times\mathbb{C}^*}(X//\mathbf{T})^a$, $\sigma\in H^*_{\mathbf{S}\times\mathbb{C}^*}(X//\mathbf{G})$, denote σ by γ/ω if $\tilde\sigma\cup\omega=\gamma$. We also denote by $\mathcal{L}^{\mathbf{S}}_{X//\mathbf{G}}$, $\mathcal{L}^{\mathbf{S}\times\mathbb{C}^*}_{X//\mathbf{G}}$, and $\mathcal{L}^{\mathbf{S}\times\mathbb{C}^*}_{\mathcal{V}_\mathbf{G}}$ the Lagrangian cones given respectively by the $\mathbf{S}$-equivariant, $\mathbf{S}\times\mathbb{C}^*$-equivariant, and $\mathcal{V}_\mathbf{G}$-twisted, $\mathbf{S}\times\mathbb{C}^*$-equivariant rational GW-invariants of $X//\mathbf{G}$.

By (the $\mathbf{S}$-equivariant version of) Lemma 5.3.1

$$\frac{z\partial_\omega J^{\mathbf{S}}_{X//\mathbf{T}}\big|_{Q^{\tilde\beta}=(-1)^{\epsilon(\tilde\beta)}Q^\beta,N'}(-z)}{\omega}\in\mathcal{L}^{\mathbf{S}}_{X//\mathbf{G}}.$$

Hence

$$\Delta_{\mathcal{V}_\mathbf{G}}\frac{z\partial_\omega J^{\mathbf{S}}_{X//\mathbf{T}}\big|_{Q^{\tilde\beta}=(-1)^{\epsilon(\tilde\beta)}Q^\beta,N'}(-z)}{\omega}\in\Delta_{\mathcal{V}_\mathbf{G}}\mathcal{L}^{\mathbf{S}\times\mathbb{C}^*}_{X//G}=\mathcal{L}^{\mathbf{S}\times\mathbb{C}^*}_{\mathcal{V}_\mathbf{G}}$$

by [CG, Corollary 4], where

$$\Delta_{\mathcal{V}_\mathbf{G}}=\prod_{\rho_i:\text{ Chern roots of }\mathcal{V}_G}b_{\rho_i}(\lambda',z),$$

$$b_\rho(\lambda',z)=\exp\left(\frac{(\lambda'+\rho)\ln(\lambda'+\rho)-(\lambda'+\rho)}{z}+\sum_{m>0}\frac{B_{2m}}{2m(2m-1)}\left(\frac{z}{\lambda'+\rho}\right)^{2m-1}\right)$$

(and B_{2m} are the Bernoulli numbers). Since

$$\Delta_{\mathcal{V}_\mathbf{G}}\frac{z\partial_\omega J^{\mathbf{S}}_{X//\mathbf{T}}\big|_{Q^{\tilde\beta}=(-1)^{\epsilon(\beta)}Q^\beta,N'}}{\omega}=\frac{z\partial_\omega\widetilde\Delta_{\mathcal{V}_\mathbf{G}}J^{\mathbf{S}}_{X//T}\big|_{Q^{\tilde\beta}=(-1)^{\epsilon(\beta)}Q^\beta,N'}}{\omega}$$

and

$$\widetilde\Delta_{\mathcal{V}_\mathbf{G}}=\Delta_{\mathcal{V}_\mathbf{T}}\pmod{\ker(\cup\omega)}$$

we conclude that

$$\frac{z\partial_\omega J^{\mathbf{S}\times\mathbb{C}^*}_{\mathcal{V}_\mathbf{T}}\big|_{Q^{\tilde\beta}=(-1)^{\epsilon(\beta)}Q^\beta,N'}(-z)}{\omega}\in\mathcal{L}^{\mathbf{S}\times\mathbb{C}^*}_{\mathcal{V}_\mathbf{G}}. \tag{6.1.1}$$

Since the J-function $J_{\mathcal{V}_{\mathbf{G}}}(-z)$ is uniquely characterized by the intersection of the Lagrangian cone $\mathcal{L}_{\mathcal{V}_{\mathbf{G}}}$ with the subspace $-z + z\mathcal{H}_-$ as in [Giv3], it follows that (6.1.1) is the J-function for P'. That is,

$$J^{\mathbf{S}\times\mathbb{C}^*}_{\mathcal{V}_{\mathbf{G}}}(t,z) = \frac{z\partial_\omega J^{\mathbf{S}\times\mathbb{C}^*}_{\mathcal{V}_{\mathbf{T}}}|_{Q^{\tilde{\beta}}=(-1)^{\epsilon(\beta)}Q^\beta, N'}}{\omega}(\tau(t),z)$$

for some unique $\tau(t)$. As in Corollary 5.3.4, the relation between τ and t is given by the expansion of the right-hand side with respect to z.

We have

$$\begin{aligned} g_{\mathcal{V}_{\mathbf{G}}}(\partial_{t_i}, \partial_{t_j}) + o(z) &= g_{\mathcal{V}_{\mathbf{G}}}(\partial_{t_i} J_{\mathcal{V}_{\mathbf{G}}}, \partial_{t_j} J_{\mathcal{V}_{\mathbf{G}}}) \\ &= g_{\mathcal{V}_{\mathbf{T}}}(z\partial_{t_i}\partial_\omega J_{\mathcal{V}_{\mathbf{T}}}, z\partial_{t_i}\partial_\omega J_{\mathcal{V}_{\mathbf{T}}}) \\ &= g_{\mathcal{V}_{\mathbf{T}}}(\partial_{\eta_i\star_{\mathcal{V}_{\mathbf{T}}}\omega}, \partial_{\eta_j\star_{\mathcal{V}_{\mathbf{T}}}\omega}) + o(z) \end{aligned}$$

where $\eta_i := \partial_{t_i}(\tau)$. We conclude that $\eta_i \star_{\mathcal{V}_{\mathbf{T}}} \omega = \gamma_i \cup \omega$, hence $\tau(t)$ coincides with the map φ. □

Remark 6.1.3. If $\mathcal{V}_{\mathbf{G}}$ and $\mathcal{V}_{\mathbf{T}}$ are generated by **S**-equivariant global sections, then $J^{\mathbf{S}}_{\mathcal{V}_{\mathbf{G}}}$ and $J^{\mathbf{S}}_{\mathcal{V}_{\mathbf{T}}}$ are well-defined without the auxiliary variable λ' (see [CG]) and hence the equality of J-functions in Theorem 6.1.2 also holds without λ'.

6.2. A simple lemma. Let X be a nonsingular projective variety with an **S**-action whose fixed points are isolated, and let Y be a connected component of the nonsingular zero locus of a regular **S**-equivariant section of a **S**-equivariant bundle E. Suppose that E is generated by **S**-equivariant global sections. Let i denote the inclusion of Y in X.

Lemma 6.2.1. *If $i^*(\tilde{t}) = t$, then $J^{\mathbf{S}}_Y(t,z)|_{Q^{\mathbf{d}}=Q^{i_*\mathbf{d}}} = i^* J^{\mathbf{S}}_E(\tilde{t},z)$ where $|_{Q^{\mathbf{d}}=Q^{i_*\mathbf{d}}}$ denotes the Novikov ring base change given by the pushforward $i_* : NE_1(Y) \to NE_1(X)$.*

Proof. For each fixed point p_i of X under the **S**-action, choose a nonzero class δ_i in $H^*_{\mathbf{S}}(X) \otimes \mathbb{C}(\lambda)$ supported near p_i, and let $\{\delta^j\}$ be the dual basis, that is, $\int^{\mathbf{S}}_{c_{\text{top}}(E)\cap[X]} \delta^i \cup \delta_j = \delta_{ij}$. Note that for nonzero $\beta \in NE_1(X)$,

$$\begin{aligned} i^* J^{\mathbf{S},\beta}_E(\tilde{t},z) &= \sum_{k\,:\,p_k\in X^{\mathbf{S}}} \frac{i^*\delta_k}{n!} \int^{\mathbf{S}}_{c_{\text{top}}(\pi_* ev^*_{n+2}E)\cap[\overline{M}_{0,n+1}(X,\beta)]^{\text{vir}}} \frac{\delta^k}{z-\psi} \prod_{i=1}^n ev^*_{1+i}(\tilde{t}) \\ &= \sum_{k\,:\,p_k\in Y^{\mathbf{S}}} \frac{i^*\delta_k}{n!} \sum_{\mathbf{d}\in NE_1(Y)\,:\,i_*\mathbf{d}=\beta} \int_{[\overline{M}_{0,n+1}(Y,\mathbf{d})]^{\text{vir}}} \frac{i^*\delta^k}{z-\psi} \prod_{i=1}^n ev^*_{1+i}(t), \end{aligned}$$

where the latter equality follows from [KKP]. Note that $i^* J^{\mathbf{S},\beta}_E(\tilde{t}) = 0$ if there is no $\mathbf{d} \in NE_1(Y)$ such that $i_*\mathbf{d} = \beta$. Since $\{i^*\delta_k\}$ and $\{i^*\delta^k\}$ form a dual pair of bases in $H^*_{\mathbf{S}}(Y)\otimes\mathbb{C}(\lambda)$ with respect to the equivariant Poincaré pairing, we are done. □

Remark 6.2.2. The above Lemma is true for the nonequivariant J-functions as well, since both sides of the identity can be specialized to $\lambda = 0$.

6.3. *J*-functions of flag manifolds of classical type. Let Y be a generalized flag manifold K/P, with K a simple complex Lie group of type B, C, or D and P a parabolic subgroup. It can be viewed as a connected component of the zero locus of a canonical section of a homogeneous bundle $\mathcal{V}_{\mathbf{G}}$ over an appropriate type A partial flag manifold $X//\mathbf{G} = Fl(k_1, \ldots, k_r, n)$. Here

$$\mathcal{V} = \begin{cases} S^2(V^*) & \text{for types } B, D \\ \wedge^2(V^*) & \text{for type } C \end{cases},$$

where V is the fundamental representation space of $GL_{k_r}(\mathbb{C})$. Note that $\mathcal{V}_{\mathbf{T}}$ is decomposable into a direct sum of line bundles (since $\mathbf{T}$-representations are completely reducible).

Let $i : Y \subset X//\mathbf{G}$ be the natural inclusion and put

$$I_{\mathcal{V}_{\mathbf{G}}} := \frac{1}{\omega}\Big(\Big(\prod_{\alpha\in\Phi_+} z\partial_\alpha\Big) I_{\mathcal{V}_{\mathbf{T}}}\Big)\Big|_{Q^{\tilde\beta}=(-1)^{\epsilon(\beta)}Q^{\beta},N'},$$

$$I_{\mathcal{V}_{\mathbf{T}}} := \sum_{\tilde\beta\in NE_1(X//\mathbf{T})} \prod_{k=1}^{\int_{\tilde\beta}\rho_i} \prod_{\rho_i:\text{ Chern roots of } \mathcal{V}_{\mathbf{T}}} (\rho_i + kz) J^{\tilde\beta}_{X//\mathbf{T}}.$$

Note that $I_{\mathcal{V}_{\mathbf{T}}}$ is a $H^*(X//\mathbf{T})$-valued series and $I_{\mathcal{V}_{\mathbf{G}}}$ is a $H^*(X//\mathbf{G})$-valued series.

Let $\mathbf{S}$ be a maximal abelian subgroup of the simple complex Lie group K. It acts on the flag manifold $Fl(k_1, \ldots, k_r, n)$ with isolated fixed points and Y is an $\mathbf{S}$-invariant submanifold. Since bundles $\mathcal{V}_{\mathbf{G}}$ and $\mathcal{V}_{\mathbf{T}}$ are generated by $\mathbf{S}$-equivariant global sections and $i^* : H^*_{\mathbf{S}}(X//\mathbf{G}) \to H^*_{\mathbf{S}}(Y)$ (as well as $i^* : H^*(X//\mathbf{G}) \to H^*(Y)$) is surjective, we obtain the following

Corollary 6.3.1. *Fix a subspace N_Y of $H^*(X//\mathbf{T})^{\mathbf{W}}$ which is a lift of $H^*(Y)$ under the composite surjection $i^* \circ (\pi^*)^{-1} \circ j^*$. The J-function of Y can be expressed as*

$$J_Y(t, z)\big|_{Q^{\mathbf{d}}=Q^{i_*\mathbf{d}}} = I_{\mathcal{V}_{\mathbf{G}}}(\tau, z) + \sum_k C^k(\tau, z) i^*\left(\frac{z\partial_{\tilde t_k} I_{\mathcal{V}_{\mathbf{G}}}(\tau, z)}{\omega}\right)$$

for some unique $C^k(\tau, z) \in N(X//\mathbf{G})[[z, \tau]]$, where $\tilde t_k$ are coordinates of N_Y.

Proof. Due to Remark 6.2.2, $J_Y = i^* J_{\mathcal{V}_{\mathbf{G}}}$. Moreover, by Remark 6.1.3,

$$J_{\mathcal{V}_{\mathbf{G}}} = \frac{z\partial_\omega}{\omega} J_{\mathcal{V}_T}\Big|_{Q^{\tilde\beta}=(-1)^{\epsilon(\beta)}Q^{\beta},N'}.$$

Now apply the quantum Lefschetz theorem of Coates and Givental [CG] and use a similar argument to the one in the proof of Theorem 5.3.3 to conclude that $i^*\left(\frac{I_{\nu_{\mathbf{G}}}(-z)}{\omega}\right)$ generates the Lagrangian cone $\mathcal{L}_Y$. □

Remark 6.3.2. This in particular reproves the result on small J-function of flag manifolds of types B, C, D in [BCK2]. No coordinate change is necessary for the explicit description of this small $J|_{t_{\text{small}}}$.

7. Appendix: Multi-point GW-invariants of Grassmannians

Recall from Sect. 4.3 the notation

$$I_{n,\beta}(\gamma_1, \dots, \gamma_n) = (-1)^{\epsilon(\beta)} \sum_{\tilde{\beta} \mapsto \beta} \langle \gamma_1, \dots, \gamma_n \rangle^{X//\mathbf{T}}_{0,n,\tilde{\beta}}.$$

Theorem 4.1.1, together with (3.5.3) (or, better, (4.2.3)), imply that Gromov–Witten invariants of a flag manifold can be written in terms of invariants of the corresponding toric variety $X//\mathbf{T}$ by a formula of the form

$$\langle \sigma_{i_1}, \dots, \sigma_{i_n} \rangle^{X//\mathbf{G}}_{0,n,\beta} = I_{n,\beta}(\tilde{\sigma}_{i_1}, \dots, \tilde{\sigma}_{i_{n-2}}, \tilde{\sigma}_{i_{n-1}} \cup \omega, \tilde{\sigma}_{i_n} \cup \omega) + \text{correction}$$

where “correction” is an expression involving invariants $I_{n',\beta'}(\dots, \tilde{\sigma}_a \cup \omega, \tilde{\sigma}_b \cup \omega)$ with $n' \le n$ and $\beta' \le \beta$. Without going into too many details, this can be seen as follows. Using the double bracket notation for derivatives of Gromov–Witten potentials mentioned in Sect. 2.1, one writes (4.2.3) as

$$\langle\langle \sigma_i, \sigma_j \rangle\rangle_{X//\mathbf{G}}(s) = \langle\langle \tilde{\sigma}_i, \tilde{\sigma}_j \rangle\rangle_{X//\mathbf{T}}(\tilde{t}(s)),$$

with $\tilde{t}(s)$ the inverse of the change of variables (3.6.2). This is an equality of power series in s-variables, and the formula for GW-invariants is obtained by identifying the coefficients of monomials in the s_j's. The coefficient of an s-monomial in the power series $\tilde{t}_k(s)$ can be explicitly expressed using the Lagrange Inversion Formula (see [GJ, Theorem 1.2.9]) in terms of the coefficients of $\tilde{t}$-monomials of *lower* total degree in the power series $s(\tilde{t})$ from (3.6.2).

The above discussion shows that the correction term will in general be quite complicated. Moreover, while it is possible in principle to give an exact expression, this will require the use of Lagrange inversion for computing the inverse $\tilde{t}(s)$ of the coordinate change (3.6.2), or, equivalently, the inverse (expressed in s-variables) of the matrix of quantum multiplication with ω on a lift of $H^*(X//\mathbf{G})$.

However, since flag manifolds are Fano of index ≥ 2, a different approach that uses Lemma 3.6.1(i) will allow us to reduce to computing only the inverse of the matrix of *small* quantum multiplication with ω. In the case of Grassmannians, when the associated abelian quotient is a product of projective spaces, it is an easy observation that the small quantum product with ω is trivial ([BCK1, Lemma 2.4]), hence no matrix inversion is necessary. We present the derivation of closed formulae for Grassmannians in this appendix.

Let $Gr := Grass(k, n)$ be the Grassmannian of k-planes in n-space, thought of as the GIT quotient $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)//GL_k(\mathbb{C})$. The abelian quotient is $\mathbb{P} := (\mathbb{P}^{n-1})^k$. We consider the usual Schubert basis $\{\sigma_\lambda\}$ of $H^*(Gr, \mathbb{C})$, indexed by partitions λ whose Young diagrams fit in a $k \times (n-k)$ rectangle. We denote by $\mathcal{P}(k, n)$ the set of all such partitions. The intersection form in this basis is given by

$$\int_{Gr} \sigma_\lambda \cup \sigma_\mu = \delta_{\mu\lambda^\vee},$$

where $\lambda^\vee$ the complementary partition to λ in the $k \times (n-k)$ rectangle. The Grassmannian has Picard number 1, so the Novikov ring is $\mathbb{C}[[Q]]$. On the other hand, the Picard group of $\mathbb{P}$ is isomorphic to $\mathbb{Z}^k$ and is generated by $H_1, \ldots, H_k$, with H_j the pull-back of the hyperplane class on the j^{th} factor. The Novikov ring of $\mathbb{P}$ is $\mathbb{C}[[Q_1, \ldots, Q_k]]$, and the specialization of Novikov variables is $Q_i = (-1)^{k-1}Q$. In this case we also have a "canonical" lifting of a class on Gr to a $\mathbf{W}$-invariant class on $\mathbb{P}$ by taking

$$\tilde{\sigma}_\lambda = S_\lambda(H_1, \ldots, H_k),$$

with S_λ the Schur polynomial of the partition λ. A curve class $\tilde{d} = (d_1, \ldots, d_k)$ on $\mathbb{P}$ is a lift of the curve class d on Gr if and only if $\sum_{i=1}^k d_i = d$. Finally, we have

$$\omega = \sqrt{\frac{(-1)^{\binom{k}{2}}}{k!}} \prod_{i<j} (H_i - H_j).$$

Let $\lambda^1, \ldots, \lambda^l$ be (not necessarily distinct) partitions. The generating function for the l-point invariants of Gr with σ_{λ^i}'s as insertions is

$$\langle\langle \sigma_{\lambda^1}, \ldots, \sigma_{\lambda^l} \rangle\rangle_{Gr}\big|_{t_{\text{small}}} = \sum_{d \geq 0} q^d \langle \sigma_{\lambda^1}, \ldots, \sigma_{\lambda^l} \rangle^{Gr}_{0,l,d},$$

where $q^d = (Qe^{t_{\text{small}}})^d$. We start with three-point invariants. Let ξ_λ be the horizontal vector field (for the connection ${}^\omega\nabla$) in Θ_N corresponding to σ_λ via the isomorphism φ of Frobenius manifolds in Theorem 4.1.1. We have (cf. (3.5.3))

$$\begin{aligned} \langle\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle\rangle_{Gr}(t) &= \xi_\lambda(\xi_\mu(\xi_\nu(F')))(\varphi(t)) \\ &= \langle\langle \hat{\xi}_\lambda, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega \rangle\rangle_{\mathbb{P}}\big|_{Q_i=(-1)^{k-1}Q, N}(\varphi(t)) \end{aligned}$$

where $\hat{\xi}_\lambda$ is an extension of ξ_λ to a vector field on M. To unburden the notation, this extension of vector fields will be understood when necessary, and the same letter will be used for a vector field in Θ_N, or its extension to Θ_M. Moreover, the specialization of Novikov variables and the restriction to N will be denoted by $\langle\langle \dots \rangle\rangle^{\bar{}}$. Hence we rewrite the last equation as

$$\text{(7.0.1)} \qquad \langle\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle\rangle_{Gr}(t) = \langle\langle \xi_\lambda, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega \rangle\rangle^{\bar{}}_{\mathbb{P}}(\varphi(t)).$$

By Lemma 3.6.1 (i) we get

$$\langle\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle\rangle_{Gr}(t)\Big|_{t_{\text{small}}} = \langle\langle \xi_\lambda, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega \rangle\rangle^{\bar{}}_{\mathbb{P}}\Big|_{\tilde{t}_{\text{small}}}.$$

From the relation $\xi_\lambda \star \omega = \tilde{\sigma}_\lambda \cup \omega$, and the fact that $\tilde{\sigma}_\lambda \star \omega|_{\tilde{t}_{\text{small}}} = \tilde{\sigma}_\lambda \cup \omega$, we obtain

$$\text{(7.0.2)} \qquad \xi_\lambda|_{\tilde{t}_{\text{small}}} = \tilde{\sigma}_\lambda.$$

It follows that

$$\begin{aligned}\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle^{Gr}_{0,3,d} &= I^{\mathbb{P}}_{3,d}(\tilde{\sigma}_\lambda, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega) \\ &= (-1)^{(k-1)d} \sum_{d_1+\cdots+d_k=d} \langle \tilde{\sigma}_\lambda, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega \rangle^{\mathbb{P}}_{0,3,(d_1,\dots,d_k)},\end{aligned}$$

an equation which was proved in [BCK1].

To obtain 4-point invariants we take the derivative of the relation (7.0.1) and get

$$\begin{aligned}\text{(7.0.3)} \quad \langle\langle \sigma_\pi, \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle\rangle_{Gr}(t) &= \xi_\pi(\langle\langle \xi_\lambda, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega \rangle\rangle^{\bar{}}_{\mathbb{P}}(\varphi(t))) \\ &= \langle\langle \xi_\pi, \xi_\lambda, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega \rangle\rangle^{\bar{}}_{\mathbb{P}}(\varphi(t)) \\ &\quad + \langle\langle \nabla_{\xi_\pi}\xi_\lambda, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega \rangle\rangle^{\bar{}}_{\mathbb{P}}(\varphi(t)),\end{aligned}$$

where $\nabla = \nabla^{\mathbb{P}}$ is the connection on M. Since

$$\begin{aligned}0 = {}^{\omega}\nabla_{\xi_\pi}\xi_\lambda \star \omega &= \nabla_{\xi_\pi}(\xi_\lambda \star \omega) \\ &= (\nabla_{\xi_\pi}\xi_\lambda) \star \omega + \sum_{a\in\mathcal{P}(k,n)} \langle\langle \xi_\pi, \xi_\lambda, \omega, \tilde{\sigma}_a \cup \omega \rangle\rangle^{\bar{}}_{\mathbb{P}}(\tilde{\sigma}_{a^\vee} \cup \omega),\end{aligned}$$

it follows that

$$\text{(7.0.4)} \qquad \nabla_{\xi_\pi}\xi_\lambda = -\sum_{a\in\mathcal{P}(k,n)} \langle\langle \xi_\pi, \xi_\lambda, \omega, \tilde{\sigma}_a \cup \omega \rangle\rangle^{\bar{}}_{\mathbb{P}}\, \xi_{a^\vee}.$$

Combining with (7.0.3) we find

$$\begin{aligned}\text{(7.0.5)}\ &\langle\langle \sigma_\pi, \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle\rangle_{Gr}(t) \\ &= \langle\langle \xi_\pi, \xi_\lambda, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega \rangle\rangle^{\bar{}}_{\mathbb{P}}(\varphi(t)) \\ &\quad - \sum_{a\in\mathcal{P}(k,n)} \langle\langle \xi_\pi, \xi_\lambda, \omega, \tilde{\sigma}_a \cup \omega \rangle\rangle^{\bar{}}_{\mathbb{P}} \langle\langle \xi_{a^\vee}, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega \rangle\rangle^{\bar{}}_{\mathbb{P}}(\varphi(t)).\end{aligned}$$

Now we restrict to t_{small}, using (7.0.2), to get

$$\begin{aligned}
&\langle \sigma_\pi, \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle^{Gr}_{0,4,d} \\
&\qquad = I^{\mathbb{P}}_{4,d}(\tilde{\sigma}_\pi, \tilde{\sigma}_\lambda, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega) \\
&\qquad\quad - \sum_{a \in \mathcal{P}(k,n)} \sum_{e+f=d} I^{\mathbb{P}}_{4,e}(\tilde{\sigma}_\pi, \tilde{\sigma}_\lambda, \omega, \tilde{\sigma}_a \cup \omega) I^{\mathbb{P}}_{3,f}(\tilde{\sigma}_{a^\vee}, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega).
\end{aligned}$$

The following remark is in order: while the left-hand side of the last formula is manifestly invariant under permutations of the indices π, λ, μ, and ν, it is not at all obvious that the right-hand side has this property. The invariance can, however, be checked directly using the splitting axiom for Gromov–Witten invariants, the vanishing result in Lemma 4.3.2, and the triviality of the small quantum product with ω.

Taking another derivative in (7.0.5) we get

$$\begin{aligned}
&\langle\langle \sigma_\rho, \sigma_\pi, \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle\rangle_{Gr}(t) \\
&\quad = \langle\langle \xi_\rho, \xi_\pi, \xi_\lambda, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega \rangle\rangle^{-}_{\mathbb{P}}(\varphi(t)) \\
&\qquad + \langle\langle \nabla_{\xi_\rho} \xi_\pi, \xi_\lambda, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega \rangle\rangle^{-}_{\mathbb{P}}(\varphi(t)) \\
&\qquad + \langle\langle \xi_\pi, \nabla_{\xi_\rho} \xi_\lambda, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega \rangle\rangle^{-}_{\mathbb{P}}(\varphi(t)) \\
&\qquad - \sum_a \big(\langle\langle \xi_\rho, \xi_\pi, \xi_\lambda, \omega, \tilde{\sigma}_a \cup \omega \rangle\rangle^{-}_{\mathbb{P}} \langle\langle \xi_{a^\vee}, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega \rangle\rangle^{-}_{\mathbb{P}} \\
&\qquad\qquad + \langle\langle \nabla_{\xi_\rho} \xi_\pi, \xi_\lambda, \omega, \tilde{\sigma}_a \cup \omega \rangle\rangle^{-}_{\mathbb{P}} \langle\langle \xi_{a^\vee}, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega \rangle\rangle^{-}_{\mathbb{P}} \\
&\qquad\qquad + \langle\langle \xi_\pi, \nabla_{\xi_\rho} \xi_\lambda, \omega, \tilde{\sigma}_a \cup \omega \rangle\rangle^{-}_{\mathbb{P}} \langle\langle \xi_{a^\vee}, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega \rangle\rangle^{-}_{\mathbb{P}} \\
&\qquad\qquad + \langle\langle \xi_\pi, \xi_\lambda, \omega, \tilde{\sigma}_a \cup \omega \rangle\rangle^{-}_{\mathbb{P}} \langle\langle \xi_\rho, \xi_{a^\vee}, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega \rangle\rangle^{-}_{\mathbb{P}} \\
&\qquad\qquad + \langle\langle \xi_\pi, \xi_\lambda, \omega, \tilde{\sigma}_a \cup \omega \rangle\rangle^{-}_{\mathbb{P}} \langle\langle \nabla_{\xi_\rho} \xi_{a^\vee}, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega \rangle\rangle^{-}_{\mathbb{P}} \big)(\varphi(t)).
\end{aligned}$$

As above, we use (7.0.4) to replace the $\nabla_{\xi_\bullet} \xi_\bullet$ insertions, then restrict to t_{small} and obtain the following formula for 5-point invariants:

$$\begin{aligned}
&\langle \sigma_\rho, \sigma_\pi, \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle^{Gr}_{0,5,d} \\
&\quad = I^{\mathbb{P}}_{5,d}(\tilde{\sigma}_\rho, \tilde{\sigma}_\pi, \tilde{\sigma}_\lambda, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega) \\
&\qquad - \sum_a \sum_{e+f=d} \big(I^{\mathbb{P}}_{5,e}(\tilde{\sigma}_\rho, \tilde{\sigma}_\pi, \tilde{\sigma}_\lambda, \omega, \tilde{\sigma}_a \cup \omega) I^{\mathbb{P}}_{3,f}(\tilde{\sigma}_{a^\vee}, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega) \\
&\qquad + I^{\mathbb{P}}_{4,e}(\tilde{\sigma}_\rho, \tilde{\sigma}_\pi, \omega, \tilde{\sigma}_a \cup \omega) I^{\mathbb{P}}_{4,f}(\tilde{\sigma}_{a^\vee}, \tilde{\sigma}_\lambda, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega) \\
&\qquad + I^{\mathbb{P}}_{4,e}(\tilde{\sigma}_\rho, \tilde{\sigma}_\lambda, \omega, \tilde{\sigma}_a \cup \omega) I^{\mathbb{P}}_{4,f}(\tilde{\sigma}_{a^\vee}, \tilde{\sigma}_\pi, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega) \\
&\qquad + I^{\mathbb{P}}_{4,e}(\tilde{\sigma}_\pi, \tilde{\sigma}_\lambda, \omega, \tilde{\sigma}_a \cup \omega) I^{\mathbb{P}}_{4,f}(\tilde{\sigma}_{a^\vee}, \tilde{\sigma}_\rho, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega) \big) \\
&\qquad + \sum_{a,b} \sum_{e+f+h=d} \big(I^{\mathbb{P}}_{4,e}(\tilde{\sigma}_\rho, \tilde{\sigma}_\pi, \omega, \tilde{\sigma}_b \cup \omega) I^{\mathbb{P}}_{4,f}(\tilde{\sigma}_{b^\vee}, \tilde{\sigma}_\lambda, \omega, \tilde{\sigma}_a \cup \omega) \\
&\qquad\qquad\qquad \times I^{\mathbb{P}}_{3,h}(\tilde{\sigma}_{a^\vee}, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega)
\end{aligned}$$

$$+ I^{\mathbb{P}}_{4,e}(\tilde{\sigma}_\rho, \tilde{\sigma}_\lambda, \omega, \tilde{\sigma}_b \cup \omega) I^{\mathbb{P}}_{4,f}(\tilde{\sigma}_{b^\vee}, \tilde{\sigma}_\pi, \omega, \tilde{\sigma}_a \cup \omega) I^{\mathbb{P}}_{3,h}$$
$$\times (\tilde{\sigma}_{a^\vee}, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega)$$
$$+ I^{\mathbb{P}}_{4,e}(\tilde{\sigma}_\pi, \tilde{\sigma}_\lambda, \omega, \tilde{\sigma}_b \cup \omega) I^{\mathbb{P}}_{4,f}(\tilde{\sigma}_{b^\vee}, \tilde{\sigma}_\rho, \omega, \tilde{\sigma}_a \cup \omega) I^{\mathbb{P}}_{3,h}$$
$$\times (\tilde{\sigma}_{a^\vee}, \tilde{\sigma}_\mu \cup \omega, \tilde{\sigma}_\nu \cup \omega)\Big).$$

It is now clear how to proceed to obtain and prove by induction a formula for Gromov–Witten invariants with an arbitrary number of insertions. We leave this to the reader.

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