

# GROMOV-WITTEN INVARIANTS FOR ABELIAN AND NONABELIAN QUOTIENTS

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## 1. Introduction.

Let  $X$  be a smooth projective variety over  $\mathbb{C}$  with the (linearized) action of a complex reductive group  $G$ , and let  $T \subset G$  be a maximal torus. In this setting, there are two geometric invariant theory (GIT) quotients,  $X//T$  and  $X//G$ , with a rational map  $\Phi : X//T \dashrightarrow X//G$  between them. We will further assume that “stable = semistable” in the GIT and that all isotropy of stable points is trivial, so  $X//T$  and  $X//G$  are smooth projective varieties, and  $\Phi$  is a  $G/T$  fibration.

Ellingsrud and Strømme [ES] and Martin [Mar] studied the relation between the intersection theory of such quotients. In particular, there is a lift of cohomology classes  $\gamma \in H^*(X//G, \mathbb{Q})$  to invariant classes  $\tilde{\gamma} \in H^*(X//T, \mathbb{Q})^W$  (for the action of the Weyl group  $W$ ), and Martin’s *integration formula* relates the Poincaré pairings:

$$\int_{X//G} \gamma \wedge \gamma' = \frac{1}{|W|} \int_{X//T} (\tilde{\gamma} \wedge \Delta^+) \wedge (\tilde{\gamma}' \wedge \Delta^-)$$

where  $\Delta^+ = \prod c_1(L_\alpha)$ , product over the positive roots  $\alpha$  (with line bundle  $L_\alpha$ ), and  $\Delta^-$  is the corresponding product for the negative roots. (For a root  $\alpha$  of  $G$ , the 1-dimensional  $T$ -representation  $\mathbb{C}_\alpha$  with weight  $\alpha$  gives rise to the  $T$ -equivariant line bundle  $X \times \mathbb{C}_\alpha$ , hence to an induced line bundle  $L_\alpha$  on the quotient  $X//T$ .)

Gromov-Witten theory generalizes the intersection theory of a smooth projective variety  $Y$  by means of intersection numbers on moduli spaces of maps from curves to  $Y$ . In [HV], Hori and Vafa made a “physics” conjecture relating Gromov-Witten theories of  $X//G = \mathbb{G}(s, n)$ , the Grassmannian of  $s$ -planes in  $\mathbb{C}^n$ , and  $X//T = \mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1} = (\mathbb{P}^{n-1})^s$  (a mathematical version of the conjecture for genus zero curves was proved in our earlier paper [BCK]), and they suggested that their conjecture should extend to general flag manifolds.

In this paper, we contend that the appropriate generalized context for the Hori-Vafa conjecture is that of nonabelian/abelian quotients described above and we state precise mathematical conjectures for the genus zero theory. In the second part of the paper, we prove the “ $J$ -function” conjecture for flag manifolds.

Given a smooth projective variety  $Y$ , a class  $d \in H_2(Y, \mathbb{Z})$ , cohomology classes  $\gamma_1, \dots, \gamma_n \in H^{2*}(Y, \mathbb{Q})$  and  $a_1, \dots, a_n \geq 0$ , then  $\langle \tau_{a_1}(\gamma_1), \dots, \tau_{a_n}(\gamma_n) \rangle_d \in \mathbb{Q}$  denotes the associated (genus zero) Gromov-Witten invariant (see §3)

When a vector bundle  $E$  on  $Y$  is given, one can also define Gromov-Witten invariants twisted by a multiplicative characteristic class of  $E$ , see [CG]. Our main conjecture (in a somewhat imprecise form, see (4.2) for the precise statement) can be viewed as a “quantum” version of Martin’s integration formula:

**Conjecture 1.1.** The genus zero Gromov-Witten invariants of  $X//G$  are expressible in terms of genus zero Gromov-Witten invariants of  $X//T$  twisted by the Euler class of the bundle  $E = \bigoplus_{\alpha \in \{\text{roots of } G\}} L_\alpha$ .

The  $n = 1$  invariants suffice for many applications to enumerative geometry. Here, a cohomology-valued formal generating function of  $(t_0, \mathbf{t}) \in H^0(Y, \mathbb{C}) \oplus H^2(Y, \mathbb{C})$  (and an additional parameter  $\hbar$ ) is formed:

$$J^Y(t_0, \mathbf{t}, \hbar) = e^{t_0 + \mathbf{t}/\hbar} \sum_d e^{\int_d \mathbf{t}} J_d^Y(\hbar),$$

with  $J_d^Y(\hbar)$  defined by  $\int_Y J_d^Y(\hbar) \wedge \gamma = \sum_{a=0}^{\infty} \hbar^{-a-2} \langle \tau_a(\gamma) \rangle_d$ .

A special case of Conjecture 1 says then that the  $J$ -function of  $X//G$  can be calculated from the  $J$ -function of  $X//T$ . Precisely, the relation is as follows: Set

$$I_d(\hbar) := \sum_{\tilde{d} \rightarrow d} \left( \prod_{\alpha} \frac{\prod_{k=-\infty}^{\tilde{d} \cdot c_1(L_\alpha)} (c_1(L_\alpha) + k\hbar)}{\prod_{k=-\infty}^0 (c_1(L_\alpha) + k\hbar)} \right) J_{\tilde{d}}^{X//T}(\hbar),$$

summed over all curve classes  $\tilde{d} \in H_2(X//T)$  lifting  $d \in H_2(X//G)$  (see (4.1)), and

$$I(t_0, \mathbf{t}, \hbar) = e^{t_0 + \mathbf{t}/\hbar} \sum_d e^{\int_d \mathbf{t}} I_d(\hbar).$$

**Conjecture 1.2.**  $J^{X//G}(t_0, \mathbf{t}, \hbar)$  is obtained from  $I$  by an explicit change of variables (“mirror transformation”). If  $X//G$  is Fano of index  $\geq 2$ ,

$$J^{X//G}(t_0, \mathbf{t}, \hbar) = I(t_0, \mathbf{t}, \hbar).$$

**Remarks:** Conjecture 2 resembles the quantum Lefschetz theorem for “concavex” bundles (sums of ample and anti-ample line bundles). Indeed, the modification to  $J_{\tilde{d}}$  has exactly the form it would have if  $\bigoplus L_\alpha$  were concavex (which it isn’t!) and  $X//G$  were a complete intersection in  $X//T$  defined by

the convex part of the bundle. See [Kim2], [Kim3] [Lee]. In fact, our most general conjectures (see (4.2) and (4.3)) include a general version of quantum Lefschetz, involving an additional twist by Euler classes of homogeneous vector bundles on  $X//G$  and  $X//T$ .

Consider the flag manifold  $F := Fl(s_1, \dots, s_l, n = s_{l+1})$  parametrizing flags:

$$\mathbb{C}^{s_1} \subset \dots \subset \mathbb{C}^{s_l} \subset \mathbb{C}^n$$

and let  $H_{i,j}$ ,  $j = 1, \dots, s_i$  be Chern roots of the duals of the universal bundles  $S_i$ :

$$S_1 \subset S_2 \subset \dots \subset S_l \subset S_{l+1} = \mathbb{C}^n \otimes \mathcal{O}_F$$

and  $d = (d_1, \dots, d_l)$  be the degree of a curve class, obtained by pairing with  $c_1(S_i^V)$ . Then Conjecture 2, together with Givental's formula for the  $J$ -function of the relevant toric variety gives the following closed formula for the  $J$ -function of  $F$  which we will prove and generalize to the other "classical" flag manifolds:

**Theorem 1.3.** *For curve classes  $d$  on the flag manifold  $F$ :*

$$J_d^F(\hbar) = \sum_{\sum d_{i,j}=d_i} \prod_{i=1}^l \left( \prod_{1 \leq j \neq j' \leq s_i} \frac{\prod_{k=-\infty}^{d_{i,j}-d_{i,j'}} (H_{i,j} - H_{i,j'} + k\hbar)}{\prod_{k=-\infty}^0 (H_{i,j} - H_{i,j'} + k\hbar)} \right. \\ \left. \prod_{1 \leq j \leq s_i, 1 \leq j' \leq s_{i+1}} \frac{\prod_{k=-\infty}^0 (H_{i,j} - H_{i+1,j'} + k\hbar)}{\prod_{k=-\infty}^{d_{i,j}-d_{i+1,j'}} (H_{i,j} - H_{i+1,j'} + k\hbar)} \right)$$

**Remark.** With the same proof as in [BCK] for Grassmannians, it follows from Theorem 1 and its generalization that Givental's  $R$ -Conjecture, hence by [Giv2] the Virasoro Conjecture, holds for classical generalized flag manifolds.

Earlier work on the  $J$ -function of type  $A$  flag manifolds is contained in the recent paper [LLY], where a formula is given for  $\int_F e^t J_d^F$ , but the problem of finding a closed formula for the function  $J$  and a generalization of the Hori-Vafa Conjecture is left open. The reader may want to compare our Theorem 1 with the formula on page 39 of [LLY]. Our proof of the Theorem is a routine verification of our conjecture using Grothendieck quot schemes. The main point here is that once one has the correct conjecture, it is an easy matter to verify it. In particular, we do not use any of the results in [LLY].

**Acknowledgments.** Thanks are due to Alexander Givental, Dosang Joe, Yuan-Pin Lee, Jae-Suk Park, Ravi Vakil and Youngho Woo for useful discussions. The final writing of this paper was done while the authors visited National Center for Theoretical Sciences in Hsinchu, Taiwan, who we thank for their hospitality and wonderful working environment. A. Bertram and I. Ciocan-Fontanine were partially supported by NSF grants DMS-0200895 and

DMS-0303614, respectively. B. Kim was supported by KOSEF 1999-2-102-003-5, R02-2002-00-00134-0.

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### 2. Classical intersection theory of GIT quotients.

#### (2.1) Abelian and non-abelian quotients.

Let  $X$  be a smooth projective variety with fixed ample line bundle  $\mathcal{L}$  and linearized action of  $G$  as in the introduction. We denote the  $G$ -stable points by  $X^s(G)$ , respectively the  $T$ -stable points by  $X^s(T)$ , so that

$$X//G = X^s(G)/G \quad \text{and} \quad X//T = X^s(T)/T$$

The main example to have in mind is the situation considered in [ES] where there is a vector space  $V$  on which  $G$  acts linearly via a representation  $G \rightarrow GL(V)$  whose image contains the homotheties of  $V$ , so that there is an induced action on  $X = \mathbb{P}(V)$ . We adopt the notation in [ES] in this case and write  $V//G$  and  $V//T$  for the quotients. Here it is clear that the unstable locus  $X - X^s(G)$  has codimension  $\geq 2$ . We will make this (mild) assumption as well in the general case.

Under our hypotheses there is a diagram relating the two quotients:

$$\begin{array}{ccc}
 U & := & X^s(G)/T \xrightarrow{i} X^s(T)/T = X//T \\
 (2.1) & & \downarrow \Phi \\
 & & X//G = X^s(G)/G
 \end{array}$$

The map  $i$  is an open immersion, while  $\Phi$  is a fibre bundle with fibre  $G/T$ , and can be further factored as

$$U \xrightarrow{e} Z \xrightarrow{\eta} X//G$$

with  $\eta$  a  $G/B$ -bundle and  $\rho$  an (affine)  $B/T$  bundle. The diagram is constructed in detail in [ES], §2.

Let  $R$  be the root system of  $G$  corresponding to the choice of the torus  $T$  and denote by  $R^+$  the set of positive roots and by  $R^-$  the set of negative roots. For each root  $\alpha \in R$  there is an induced line bundle  $L_\alpha$  on  $X//T$ , coming from the canonical 1-dimensional representation  $\mathbb{C}_\alpha$  of  $T$  with weight  $\alpha$ . Precisely,  $L_\alpha = \mathbb{C}_\alpha \times_T X^s(T)$ . If  $(\alpha, -\alpha)$  is a pair of opposite roots, then the corresponding line bundles are dual, and  $c_1(L_\alpha) = -c_1(L_{-\alpha})$ .

We denote

$$E^+ := \bigoplus_{\alpha \in R^+} L_\alpha, \quad E^- := \bigoplus_{\alpha \in R^-} L_\alpha, \quad E := E^+ \oplus E^-.$$

The Euler classes

$$\Delta = \Delta^+ = \text{Euler}(E^+) = \prod_{\alpha \in R^+} c_1(L_\alpha)$$

and

$$\Delta^- = (-1)^{\dim(G/B)} \Delta = \text{Euler}(E^-) = \prod_{\alpha \in R^-} c_1(L_\alpha)$$

will play an important role in this paper.

Note that the Weyl group  $W = N(T)/T$  of  $G$  acts naturally on  $X//T$ , and therefore on the cohomology  $H^*(X//T, \mathbb{Q})$ . The classes  $\Delta$  and  $\Delta^-$  are  $W$ -anti-invariant, i.e. if  $w$  is an element of  $W$  of length  $\ell(w)$ , then  $w(\Delta) = (-1)^{\ell(w)} \Delta$ .

## (2.2) Cohomology of $X//G$ versus cohomology of $X//T$ .

Unless mentioned otherwise, we will only consider cohomology with  $\mathbb{Q}$  coefficients. We recall some results of Ellingsrud-Strømme [ES] and Martin [Mar] relating the cohomology rings  $H^*(X//G)$  and  $H^*(X//T)$ .

There are surjective Kirwan maps

$$\kappa_G : H_G^*(X) \longrightarrow H^*(X//G) \quad \text{and} \quad \kappa_T : H_T^*(X) \longrightarrow H^*(X//T)$$

from equivariant cohomology of  $X$  to the cohomology of the quotients, as well as a natural restriction map

$$\tau_T^G : H_G^*(X) \longrightarrow H_T^*(X).$$

For cohomology classes  $\gamma$  on  $X//G$  and  $\tilde{\gamma}$  on  $X//T$  we say that  $\tilde{\gamma}$  is a *lift* of  $\gamma$  if they come from the same  $G$ -equivariant class on  $X$ :

$$\gamma = \kappa_G(\theta), \quad \tilde{\gamma} = \kappa_T(\tau_T^G(\theta))$$

for some  $\theta \in H_G^*(X)$ . From surjectivity of  $\kappa_G$  it is clear that each  $\gamma \in H^*(X//G)$  has a lift. Equivalently, one may define the notion of lift using the maps in the basic diagram (2.1) by the requirement that

$$i^*(\tilde{\gamma}) = \Phi^*(\gamma).$$

This second description shows that  $\tilde{\gamma}$  may be taken in  $H^*(X//T)^W$ . We repeat, for emphasis, Martin's *integration formula*:

**(2.2.1) Theorem ([Mar], Theorem B).** *With notation as above,*

$$\int_{X//G} \gamma = \frac{1}{|W|} \int_{X//T} \tilde{\gamma} \wedge \Delta^+ \wedge \Delta^-$$

for any  $\gamma \in H^*(X//G)$ .

### 3. Genus zero Gromov-Witten theory.

#### (3.1) Gromov-Witten invariants with descendants

Let  $Y$  be a smooth projective variety and let  $d \in H_2(Y, \mathbb{Z})$  be fixed curve class. The Kontsevich-Manin moduli stack  $\overline{M}_{0,n}(Y, d)$  of stable maps of class  $d$  from  $n$ -pointed nodal rational curves to  $Y$  comes with  $n$  evaluation maps  $ev_1, \dots, ev_n$  to  $Y$  (at the marked points). The natural projection

$$\pi : \overline{M}_{0,n+1}(Y, d) \longrightarrow \overline{M}_{0,n}(Y, d)$$

given by forgetting the last marked point allows us to view  $\overline{M}_{0,n+1}(Y, d)$  as the universal curve. The map  $\pi$  has  $n$  sections  $s_1, \dots, s_n$  (corresponding to the marked points) defining the *Witten cotangent line bundles*  $\mathcal{L}_i := s_i^*(\omega_\pi)$ , with  $\omega_\pi$  the relative dualizing sheaf. It is customary to denote by  $\psi_i$  the Chern class  $c_1(\mathcal{L}_i)$ .

Given cohomology classes  $\gamma_1, \dots, \gamma_n \in H^{2*}(Y)$  and nonnegative integers  $a_1, \dots, a_n$ , the associated genus zero Gromov-Witten invariant is

$$\langle \tau_{a_1}(\gamma_1), \dots, \tau_{a_n}(\gamma_n) \rangle_d := \int_{[\overline{M}_{0,n}(Y, d)]^{\text{virt}}} \wedge_{i=1}^n (\psi_i^{a_i} \wedge ev_i^*(\gamma_i)),$$

where  $[\overline{M}_{0,n}(Y, d)]^{\text{virt}}$  is the virtual fundamental class of [LT], [BF]. This virtual class lives in the Chow group  $A_{\text{vdim}(\overline{M}_{0,n}(Y, d))}(\overline{M}_{0,n}(Y, d))$ , with

$$\text{vdim}(\overline{M}_{0,n}(Y, d)) = \int_d c_1(\mathcal{T}_Y) + \dim_{\mathbb{C}}(Y) + n - 3$$

the virtual dimension of  $\overline{M}_{0,n}(Y, d)$ , as given by the Riemann-Roch theorem. The invariant is in general a rational number, and it vanishes unless

$$\sum_{i=1}^n \frac{1}{2} \deg(\gamma_i) + a_i$$

equals the virtual dimension.

**(3.2) Gromov-Witten invariants twisted by the Euler class.** Assume now that on  $Y$  we are given a vector bundle  $E$ . From:

$$\begin{array}{ccc} \overline{M}_{0,n+1}(Y, d) & \xrightarrow{e} & Y \\ \downarrow \pi & & \\ \overline{M}_{0,n}(Y, d) & & \end{array}$$

with  $e = ev_{n+1}$ , we obtain

$$E_{n,d} = [R^0 \pi_* e^* E] - [R^1 \pi_* e^* E]$$

the push-forward of  $e^* E$  in  $K$ -theory. Its virtual rank is given by the Riemann-Roch formula:

$$\text{vrk}(E_{n,d}) := \text{rk}(E) + \int_d c_1(E).$$

Since  $E_{n,d}$  can be obtained as the cohomology of a 2-term complex of vector bundles on  $\overline{M}_{0,n}(Y, d)$  (see [CG]), it has a well defined “top Chern class”  $c_{\text{top}}(E_{n,d}) := c_{\text{vrk}}(E_{n,d})$ . It is of course zero if the virtual rank is negative, but this will not be the case for any of the twisting bundles we employ.

We define the Gromov-Witten invariants of  $Y$  twisted by the Euler class of  $E$  by

$$\langle \tau_{a_1}(\gamma_1), \dots, \tau_{a_n}(\gamma_n) \rangle_{d,E} := \int_{[\overline{M}_{0,n}(Y,d)]^{\text{virt}}} \wedge_{i=1}^n (\psi_i^{a_i} \wedge ev_i^*(\gamma_i)) \wedge c_{\text{top}}(E_{n,d}).$$

These twisted invariants (in much greater generality) were studied by Coates and Givental in [CG]. Their definition of twisting by the Euler class is formulated in a more general setting (essentially “twisting by the Chern polynomial”) but can be seen to specialize to the one given here for the cases we consider.

#### 4. Conjectural relations between $X//T$ and $X//G$

Recall that on  $X//T$  we have the bundle  $E = \bigoplus_{\alpha \in R} L_\alpha$ . Martin’s integration formula (2.2.1) may be viewed as expressing degree zero Gromov-Witten invariants (i.e. usual intersection numbers) on  $X//G$  in terms of degree zero invariants on  $X//T$  twisted by the Euler class of the bundle  $E$ . We conjecture that this relation extends to stable maps of higher degrees.

**(4.1) Definition.** For curve classes  $d \in H_2(X//G, \mathbb{Z})$  and  $\tilde{d} \in H_2(X//T, \mathbb{Z})$  we say that  $\tilde{d}$  lifts  $d$  (and write  $\tilde{d} \mapsto d$ ) if

$$\int_d H = \int_{\tilde{d}} \tilde{H}$$

for every divisor class  $H \in H^2(X//G, \mathbb{Q})$  with lift  $\tilde{H} \in H^2(X//G, \mathbb{Q})^W$ .

Since any two lifts agree when restricted to the “ $G$ -stable locus”  $U \subset X//T$ , and by assumption (see (2.1)) the complement of  $U$  has codimension at least 2, it follows that for divisor classes the  $W$ -invariant lifts are unique, and the  $\tilde{d}$ ’s are indeed well-defined. (Note also that by taking  $H$  to be ample in (4.1), with ample  $W$ -invariant lift, we see that each  $d$  has finitely many lifts.)

As shown by the examples we treat in §6, it is actually useful to introduce an additional twisting. Namely, consider a finite dimensional linear representation  $\mathcal{V}$  of  $G$ . It induces the homogeneous vector bundle  $\mathcal{V}_G := X^s(G) \times_G \mathcal{V}$  over  $X//G$  and, viewing  $\mathcal{V}$  as a  $T$ -representation, the vector bundle  $\mathcal{V}_T := X^s(T) \times_T \mathcal{V}$  over  $X//T$ . Since a  $T$ -representation is completely reducible,  $\mathcal{V}_T$  splits as a direct sum of line bundles, which we will assume to be nef. Note that the Euler class of  $\mathcal{V}_T$  is a lift of the Euler class of  $\mathcal{V}_G$ . With this definitions we can state:

**(4.2) Conjecture.** Twisted genus zero Gromov-Witten invariants of  $X//G$  and of  $X//T$  are related by

$$\langle \tau_{a_1}(\gamma_1), \dots, \tau_{a_n}(\gamma_n) \rangle_{d, \mathcal{V}_G} = \frac{1}{|W|} \sum_{\tilde{d} \mapsto d} \langle \tau_{a_1}(\tilde{\gamma}_1), \dots, \tau_{a_n}(\tilde{\gamma}_n) \rangle_{\tilde{d}, E \oplus \mathcal{V}_T},$$

where  $\tilde{\gamma}_i$  are lifts of  $\gamma_i$ .

In particular, when the extra twist by  $\mathcal{V}$  is absent, we are expressing the GW-invariants of  $X//G$  in terms of invariants of  $X//T$  twisted by the Euler class of  $E$ , specializing to Martin’s formula in degree zero. Next, we express this relationship in terms of Givental’s  $J$ -functions.

By the work of Coates and Givental ([CG], Thm. 2 and Cor. 5), one can extract from (4.2) a conjectural “Quantum Lefschetz formula” calculating the  $J$ -function (on the “big” parameter space  $H^{2*}(X//G)!$ ) of the nonabelian quotient  $X//G$  (or, more generally, the  $J$ -function of the theory on  $X//G$  twisted by  $\mathcal{V}_G$ ) in terms of the bundle  $E$  (respectively, the bundle  $E \oplus \mathcal{V}_T$ ) and the  $J$ -function of the abelian quotient  $X//T$ . We state explicitly a weaker version involving the restriction of  $J$  to the “small” parameter space  $H^2(X//G)$ . This restriction amounts to considering (4.2) only for 1-point invariants.



We let  $(t_0, \mathbf{t})$  denote a general element of  $H^0(X//G, \mathbb{C}) \oplus H^2(X//G, \mathbb{C})$ . The  $\mathcal{V}_G$ -twisted  $J$ -function of  $X//G$  is

$$J^{X//G, \mathcal{V}}(t_0, \mathbf{t}, \hbar) := e^{t_0 + \mathbf{t}/\hbar} \sum_d e^{\int_d \mathbf{t}} J_d^{X//G, \mathcal{V}_G}(\hbar),$$

with  $J_d^{X//G, \mathcal{V}_G}(\hbar)$  defined by

$$\int_{X//G} J_d^{X//G, \mathcal{V}}(\hbar) \wedge \gamma \wedge c_{\text{top}}(\mathcal{V}_G) = \sum_{a=0}^{\infty} \hbar^{-a-2} \langle \tau_a(\gamma) \rangle_{d, \mathcal{V}_G}.$$

Let  $M_1, \dots, M_r$  ( $r$  is the dimension of the representation  $\mathcal{V}$ ) denote the line bundle direct summands of the split bundle  $\mathcal{V}_T$  on  $X//T$  and assume all the  $M_i$  are nef, so that  $\tilde{d} \cdot c_1(M_i) \geq 0$  for every effective curve class  $\tilde{d}$ . Define

$$I^{X//T, \mathcal{V}}(t_0, \mathbf{t}, \hbar) = e^{t_0 + \mathbf{t}/\hbar} \sum_d e^{\int_d \mathbf{t}} \sum_{\tilde{d} \rightarrow d} I_{\tilde{d}}(\hbar),$$

where

$$I_{\tilde{d}}(\hbar) = \prod_{\alpha} \frac{\prod_{k=-\infty}^{\tilde{d} \cdot c_1(L_{\alpha})} (c_1(L_{\alpha}) + k\hbar)}{\prod_{k=-\infty}^0 (c_1(L_{\alpha}) + k\hbar)} \prod_{i=1}^r \frac{\prod_{k=-\infty}^{\tilde{d} \cdot c_1(M_i)} (c_1(M_i) + k\hbar)}{\prod_{k=-\infty}^0 (c_1(M_i) + k\hbar)} J_{\tilde{d}}^{X//T}(\hbar).$$

In other words,  $I^{X//T, \mathcal{V}}$  is obtained from the (untwisted)  $J$  function of  $X//T$  by first introducing in each  $J_{\tilde{d}}$  the correcting classes determined by the bundles  $E$  and  $\mathcal{V}_T$ , and then specializing the parameter  $\mathbf{t}$  to the subspace  $H^2(X//T, \mathbb{C})^W$ , which we identify with  $H^2(X//G, \mathbb{C})$ .

**(4.2) Conjecture.** *There is an explicit change of variable  $(t_0, \mathbf{t}) \mapsto f(t_0, \mathbf{t})$  such that*

$$J^{X//G, \mathcal{V}}(t_0, \mathbf{t}, \hbar) = I^{X//T, \mathcal{V}}(f(t_0, \mathbf{t}), \hbar).$$

**(4.3) Remark.** As in the usual quantum Lefschetz hyperplane theorem ([Giv1], [Kim2], [Lee], [CG]), the change of variable is read off the asymptotics of the  $1/\hbar$  expansion of the function  $I$ . In particular, if  $X//T$  is Fano, and for every effective curve  $C \subset X//T$  the intersection number  $C \cdot (c_1(X//T) - \sum_{i=1}^r c_1(M_i))$  is at least 2, then no change of variable is needed and we have the equality

$$J^{X//G, \mathcal{V}}(t_0, \mathbf{t}, \hbar) = I^{X//T, \mathcal{V}}(t_0, \mathbf{t}, \hbar).$$

This will be the case for all examples we treat in the rest of the paper.

Note that in general no analogue of the quantum Lefschetz “correcting” class is known when twisting by an indecomposable vector bundle.

### 5. Standard flag manifolds

(5.1) **The flag manifold as a GIT quotient.**  $F = Fl(s_1, \dots, s_l, n = s_{l+1})$  from the introduction is the GIT quotient:

$$F = \mathbb{P}(\oplus_{i=1}^l \text{Hom}(\mathbb{C}^{s_i}, \mathbb{C}^{s_{i+1}})) // G$$

by the action of  $G = \prod_{i=1}^l \text{GL}(s_i, \mathbb{C})$ , where a matrix  $A \in \text{GL}(s_i, \mathbb{C})$  acts on  $\text{Hom}(\mathbb{C}^{s_i}, \mathbb{C}^{s_{i+1}})$  by left multiplication, and on  $\text{Hom}(\mathbb{C}^{s_{i-1}}, \mathbb{C}^{s_i})$  by right multiplication by  $A^{-1}$ . Stability here is the ordinary stability for Grassmannians: an element  $x \in \mathbb{P}(\oplus_{i=1}^l \text{Hom}(\mathbb{C}^{s_i}, \mathbb{C}^{s_{i+1}}))$  is stable if each of its “coordinates”  $x_i \in \text{Hom}(\mathbb{C}^{s_i}, \mathbb{C}^{s_{i+1}})$  is injective. If  $T \subset G$  is the product of the subgroups of diagonal matrices, then the associated abelian quotient

$$Y := \mathbb{P}(\oplus_{j=1}^l \text{Hom}(\mathbb{C}^{s_j}, \mathbb{C}^{s_{j+1}})) // T$$

is a toric variety. Corresponding to the description of the flag manifold as a tower of Grassmannian bundles:

$$\begin{array}{ccc} \mathbb{G}(s_i, s_{i+1}) & \rightarrow & Fl(s_i, s_{i+1}, \dots, s_{l+1}) = \mathbb{G}(s_i, S_{i+1}) \\ & & \downarrow \\ & & Fl(s_{i+1}, \dots, s_{l+1}) \end{array}$$

the toric variety  $Y$  is a tower of product-of-projective-space bundles:

$$\begin{array}{ccc} \prod^{s_i} \mathbb{P}^{s_{i+1}-1} & \rightarrow & Y_i = \mathbb{P}(V_{i+1}) \times_{Y_{i+1}} \cdots \times_{Y_{i+1}} \mathbb{P}(V_{i+1}) \\ & & \downarrow \\ & & Y_{i+1} \end{array}$$

with

$$V_{i+1} = \bigoplus_{j=1}^{s_{i+1}} \mathcal{O}_{Y_{i+1}}(0, \dots, 0, \underbrace{-1, 0, \dots, 0}_j)$$

the vector bundle on  $Y_{i+1}$  corresponding to  $S_{i+1}$ , with  $Y_{l+1} = \text{Spec}(\mathbb{C})$  and  $Y_1 = Y$ .

There is the additional right action of  $\text{GL}(n, \mathbb{C})$  commuting with the action of  $G$ , descending to a (transitive) action on  $F$ , exhibiting  $F$  as a homogeneous space. When we speak of the equivariant cohomology of  $F$  and of  $Y$ , it will be with respect to the induced action of the maximal torus  $T_n \subset \text{GL}(n, \mathbb{C})$ .

The Chern roots  $H_{i,j}$ ,  $1 \leq i \leq l$ ,  $1 \leq j \leq s_i$  of the introduction can now be viewed as Chern classes of the relative hyperplane classes of  $Y_i \rightarrow Y_{i+1}$ . This way of representing the Chern roots is less efficient than the Chern classes on the full flag variety  $Fl(1, 2, \dots, n)$ , since in that case  $H_{i,j} = H_{i+1,j}$  for each  $j = 1, \dots, s_i$ , whereas in this representation the  $H_{i,j}$  are all distinct. The additional classes  $H_{l+1,j}$  really appear extraneous, as they are Chern roots

of the trivial bundle. But they play an important role in the equivariant  $J$ -function of the toric variety, which we now describe, following Givental [Giv1].

A toric variety  $Y$  comes with a finite set  $\{D_i\}$  of torus-invariant divisor classes that generate  $H^2(Y)$  additively and  $H^*(Y)$  multiplicatively. In the description of  $Y$  as a quotient  $\mathbb{C}^N//T$  for  $T \subset (\mathbb{C}^*)^n$ , these are simply the first Chern classes  $c_1(L_i)$ , where  $L_i$  are the  $N$  line bundles determined by the coordinate lines of  $\mathbb{C}^N$ . Moreover, if  $T \subset T' \subset (\mathbb{C}^*)^n$ , then the  $T'/T$ -equivariant cohomology ring of  $Y$  is similarly generated by the equivariant divisor classes determined by the coordinate lines.

**(5.1.1) Theorem (Givental, [Giv1]).** *If  $Y$  is a smooth toric variety with the property that each curve class  $d \in H_2(Y)$  satisfies  $-\int_d K_Y \geq 2$ , then the  $J$ -function of  $Y$  is given by the formula:*

$$J_d^Y(\hbar) = \prod_{i=1}^N \frac{\prod_{k=-\infty}^0 (D_i + k\hbar)}{\prod_{k=-\infty}^{\int_d D_i} (D_i + k\hbar)}$$

and the  $T'/T$ -equivariant  $J$ -function is given by the same formula, with equivariant divisor classes.

**Note:** In case  $Y = \mathbb{P}^n$  or a product of projective spaces each  $\int_d D_i \geq 0$ , and the numerator can be factored out of the denominator, but in the general case this is the most convenient formulation of the  $J$ -function.

For our toric variety, it is easy to see that the invariant divisors are:

$$H_{i,j} - H_{i+1,j'}, \quad i = 1, \dots, l-1$$

in both the ordinary and equivariant case, and in addition  $H_{l,j} - H_{l+1,j'}$  which are equivariant if  $H_{l+1,j'} = \lambda_{j'}$  in  $H^*(BT_n) = \mathbb{Q}[\lambda_1, \dots, \lambda_n]$  and ordinary if  $H_{l+1,j'} = 0$ .

Thus, by Givental's theorem, we have:

$$J_{\vec{d}}^Y(\hbar) = \prod_{i=1}^l \prod_{1 \leq j \leq s_i, 1 \leq j' \leq s_{i+1}} \frac{\prod_{k=-\infty}^0 (H_{i,j} - H_{i+1,j'} + k\hbar)}{\prod_{k=-\infty}^{d_{i,j} - d_{i+1,j'}} (H_{i,j} - H_{i+1,j'} + k\hbar)}$$

where

$$\vec{d} = (d_{1,1}, \dots, d_{1,s_1}, d_{2,1}, \dots, d_{2,s_2}, \dots, d_{l,1}, \dots, d_{l,s_l})$$

is the general curve class (with, evidently, each  $d_{l+1,j'} = 0$ ).

It is also easy to see that the roots of  $G$  give the divisor classes:

$$H_{i,j} - H_{i,j'}, \quad i = 1, \dots, l, \quad j \neq j'$$

so that our  $J$ -function conjecture becomes precisely Theorem 1. Again, we repeat for emphasis that the  $T_n$ -equivariant  $J$ -function is obtained by setting

each  $H_{l+1,j'} = \lambda_{j'}$  and the “ordinary”  $J$ -function is obtained by setting each  $H_{l+1,j'} = 0$ .

**(5.2) Conversion of the Formula.** In our previous paper [BCK], we proved that the  $J$ -function of the Grassmannian  $\mathbb{G}(s, n)$  is given by:

$$J^{\mathbb{G}}(q, \hbar) = \sum q^d J_d^{\mathbb{G}}(q, \hbar)$$

where

$$J_d^{\mathbb{G}}(\hbar) = (-1)^{(s-1)d} \sum_{\substack{(d_1, \dots, d_s) \\ d_1 + \dots + d_s = d}} \frac{\prod_{1 \leq j < j' \leq s} (H_j - H_{j'} + (d_j - d_{j'})\hbar)}{\prod_{1 \leq j < j' \leq s} (H_j - H_{j'}) \prod_{j=1}^s \prod_{k=1}^{d_j} (H_j + k\hbar)^n}$$

and  $H_i$  are Chern roots of  $S^\vee$ , the dual of the universal subbundle. We first rewrite the formula in a version more parallel to the Givental formula for toric varieties:

$$\sum_{\substack{(d_1, \dots, d_s) \\ d_1 + \dots + d_s = d}} \prod_{1 \leq j \neq j' \leq s} \frac{\prod_{k=-\infty}^{d_j - d_{j'}} (H_j - H_{j'} + k\hbar)}{\prod_{k=-\infty}^0 (H_j - H_{j'} + k\hbar)} \cdot \frac{1}{\prod_{j=1}^s \prod_{k=1}^{d_j} (H_j + k\hbar)^n}$$

and then it follows that the  $J$ -function for a *product* of Grassmannians:  $\prod \mathbb{G} = \prod_{i=1}^l \mathbb{G}(s_i, n)$  is given by:

$$(5.2.1) \quad J_{(d_1, \dots, d_l)}^{\prod \mathbb{G}}(\hbar) = \sum_{\vec{d}} \prod_{i=1}^l \left( \prod_{1 \leq j \neq j' \leq s_i} \frac{\prod_{k=-\infty}^{d_{i,j} - d_{i,j'}} (H_{i,j} - H_{i,j'} + k\hbar)}{\prod_{k=-\infty}^0 (H_{i,j} - H_{i,j'} + k\hbar)} \times \frac{1}{\prod_{j=1}^{s_i} \prod_{k=1}^{d_{i,j}} (H_{i,j} + k\hbar)^n} \right)$$

where  $\vec{d}$  is defined just as in the toric formula, with  $\sum_{j=1}^{s_i} d_{i,j} = d_i$ . Let  $S_i$  and  $Q_i = \mathbb{C}^n \otimes \mathcal{O}/S_i$  be the universal sub and quotient  $G$ -bundles, thought of either on  $\mathbb{G}(s_i, n)$ ,  $\prod \mathbb{G}(s_i, n)$  or  $F$ . Then  $F \subset \prod_{i=1}^l \mathbb{G}(s_i, n)$  is a transverse zero section of the bundle  $\mathcal{V} = \bigoplus_{i=1}^{l-1} \mathcal{H}om(S_i, Q_{i+1})$ . Of course, we may write:

$$0 \rightarrow \bigoplus_{i=1}^{l-1} \mathcal{H}om(S_i, S_{i+1}) \rightarrow \bigoplus_{i=1}^{l-1} \mathcal{H}om(S_i, \mathbb{C}^n) \rightarrow \mathcal{V} \rightarrow 0$$

and then the correction to the  $d$ th term of the  $J$ -function of  $\prod \mathbb{G}$  coming from twisting by  $\mathcal{V}$ , as predicted by our general conjecture (4.2), is:

$$(5.2.2) \quad \prod_{i=1}^{l-1} \left( \prod_{1 \leq j \leq s_i, 1 \leq j' \leq s_{i+1}} \frac{\prod_{k=-\infty}^0 (H_{i,j} - H_{i+1,j'} + k\hbar)}{\prod_{k=-\infty}^{d_{i,j} - d_{i+1,j'}} (H_{i,j} - H_{i+1,j'} + k\hbar)} \times \prod_{j=1}^{s_i} \prod_{k=1}^{d_{i,j}} (H_{i,j} + k\hbar)^n \right)$$

Or, in other words, our conjecture gives the same result whether we regard the flag variety itself as a GIT quotient, or we think of it as the zero locus of a section of  $\mathcal{V}$  in  $\coprod \mathbb{G}$ , regarded as a GIT quotient. This is not a particularly deep check of the conjecture, but it is the latter point of view that we will use in the next section to prove the Theorem, as it generalizes immediately to the other classical Lie types.

### 6. Proof of Theorem 1 and its Generalizations

**(6.1) A Simple J-function Lemma** The degree  $d$  component of the ( $T$ -equivariant)  $J$ -function of  $Y$  (with action of  $T$ ) is given by the push-forward:

$$J_d^Y(\hbar) = e_* \left( \frac{[M_d^Y]}{[M_d^Y/G_d^Y]} \right) = e_* \left( \frac{[M_d^Y]}{\hbar(\hbar - \psi)} \right)$$

in ( $T$ -equivariant) cohomology, where we use the following conventions:

- $M_d^Y = \overline{M}_{0,1}(Y, d)$  with virtual fundamental class  $[M_d^Y]$
- $G_d^Y = \overline{M}_{0,0}(Y \times \mathbb{P}^1, (d, 1))$  with virtual fundamental class  $[G_d^Y]$ .
- $\mathbb{C}^*$  acts on  $\mathbb{P}^1$  by scaling  $(x, y) \mapsto (tx, y)$  and  $H^*(B\mathbb{C}^*) = \mathbb{Q}[\hbar]$ .
- Whenever  $F \subset X$  is a fixed locus for an action of  $\mathbb{C}^*$ , then  $[F/X]$  denotes the Euler class of the normal bundle of  $F$  in  $X$  in  $H_{\mathbb{C}^*}^*(F) = H^*(F)[\hbar]$  (or  $H_T^*(F)[\hbar]$ ), which is always invertible, by the Atiyah-Bott localization theorem.

•  $M_d^Y \subset G_d^Y$  is one of the fixed components for the induced  $\mathbb{C}^*$  action on  $G_d^Y$ . Specifically, it consists of stable maps of curves with one component of class  $(0, 1)$  and the rest of the curve mapping to  $0 \in \mathbb{P}^1$ .

It is a standard fact in Gromov-Witten theory (see e.g. [Ber], [Lee]) that:

$$[M_d^Y/G_d^Y] = \hbar(\hbar - \psi)$$

When  $Y \subset \mathbb{P}^n$ , there is an equivariant “map to the linear sigma model” and diagram of fixed components:

$$\begin{array}{ccccccc} u : & G_d^Y & \rightarrow & Y_d & \subset & \mathbb{P}_d^n & = \mathbb{P}(\text{Hom}_d(\mathbb{C}^2, \mathbb{C}^{n+1})) \\ & \cup & & \cup & & \cup & \\ & M_d^Y & \xrightarrow{e} & Y & \subset & \mathbb{P}^n & \end{array}$$

where  $Y_d \subset \mathbb{P}_d^n$  is defined by the equations induced from the equations of  $Y$ , and  $\mathbb{P}^n \subset \mathbb{P}_d^n$  is the fixed locus arising from the “zero” line  $\mathbb{C} \subset \mathbb{C}^2$  and

inclusion  $\mathbb{P}^n = \mathbb{P}(\mathrm{Hom}_d(\mathbb{C}, \mathbb{C}^{n+1})) \subset \mathbb{P}_d^n$ . Set-theoretically,

$$Y_d = \coprod_{e=0}^d \mathrm{Map}_e(\mathbb{P}^1, Y) \times \mathbb{P}^{d-e},$$

so that if  $Y$  is homogeneous (which will be our case), then  $Y_d$  is the (singular!) closure of the (smooth) Hilbert scheme  $\mathrm{Map}_d(\mathbb{P}^1, Y) \subset \mathrm{Map}_d(\mathbb{P}^1, \mathbb{P}^n) \subset \mathbb{P}_d^n$  which is birational to the smooth compactification  $G_d^Y$ , and  $u_*[G_d^Y] = [Y_d]$ . (Indeed, when  $Y$  is homogeneous,  $G_d^Y$  is the (disjoint) union of  $G_\beta^Y$  over  $\beta \in H_2(Y, \mathbb{Z})$  with  $j_*\beta = d$  and each  $G_\beta^Y$  is a smooth irreducible Deligne-Mumford stack containing  $\mathrm{Map}_\beta(\mathbb{P}^1, Y)$  as a dense open substack. The restriction to  $\mathrm{Map}_d(\mathbb{P}^1, Y) = \coprod_\beta \mathrm{Map}_\beta(\mathbb{P}^1, Y)$  of the surjection  $u : G_d^Y \rightarrow Y_d$  is an isomorphism.)

Now let

$$i : X \xrightarrow{k} Y \xrightarrow{j} \mathbb{P}^n$$

be  $T$ -equivariant embeddings for an action of  $T$  on  $\mathbb{P}^n$  with isolated fixed points. Suppose  $X$  and  $Y$  are both homogeneous and  $\mathrm{Map}_d(\mathbb{P}^1, Y) \subset Q_d^Y$  is another smooth compactification with extended  $\mathbb{C}^* \times T$  action and equivariant map  $v : Q_d^Y \rightarrow \mathbb{P}_d^n$ , and suppose there is an equivariant class  $[Q_d^X] \in A_*(Q_d^Y)$  such that:

$$(\dagger) \quad v_*([Q_d^X]) = [X_d] = u_*([G_d^X])$$

Let  $\alpha_F : F \hookrightarrow Q_d^Y$  be the union of fixed components mapping to  $\mathbb{P}^n$  by  $v$ . For ease of notation, we will pretend there is only one component, writing, for example,  $\alpha_F^*[Q_d^X]/[F/Q_d^Y]$ , when we really mean the sum  $\sum \alpha_{F_k}^*[Q_d^X]/[F_k/Q_d^Y]$  over the components  $F_k \subset F$ . Let  $f : F \rightarrow \mathbb{P}^n$  be the restriction of  $v$ . It follows that  $f$  factors through a map  $g : F \rightarrow Y$ , and we get the following diagram:

$$\begin{array}{ccccc} G_d^X & \xrightarrow{u} & \mathbb{P}_d^n & \xleftarrow{v} & Q_d^Y \\ & & \alpha_d \uparrow & & \\ \alpha_d^X \uparrow & & \mathbb{P}^n & & \alpha_F \uparrow \\ & & j \uparrow & \swarrow f & \\ M_d^X & \rightarrow & Y & \xleftarrow{g} & F \\ & & e \searrow & & \\ & & k \uparrow & & \\ & & X & & \end{array}$$

**(6.1.1) Lemma.**  $[Q_d^X]$  computes  $J_d^X$ . First, in  $\mathbb{C}^*$ -equivariant cohomology:

$$i_* J_d^X = \frac{\alpha_d^* u_*([G_d^X])}{[\mathbb{P}^n/\mathbb{P}_d^n]} = \frac{\alpha_d^* v_*([Q_d^X])}{[\mathbb{P}^n/\mathbb{P}_d^n]} = f_* \frac{\alpha_F^*[Q_d^X]}{[F/Q_d^Y]} = j_* g_* \frac{\alpha_F^*[Q_d^X]}{[F/Q_d^Y]}$$

Next, in  $\mathbb{C}^* \times T$ -equivariant cohomology:

$$J_d^X = \frac{i^* i_* J_X^d}{[X/\mathbb{P}^n]} = \frac{1}{[X/\mathbb{P}^n]} i^* j_* g_* \frac{\alpha_F^*[Q_d^X]}{[F/Q_d^Y]} = \frac{1}{[X/Y]} k^* g_* \left( \frac{\alpha_F^*[Q_d^X]}{[F/Q_d^Y]} \right)$$

Finally, if  $E$  is a  $T$ -equivariant vector bundle on  $Y$ , and  $X$  is the zero scheme of a section of  $E$  transverse to the zero section, then  $[X/Y] = k^* c_{\text{top}}(E)$ , and:

$$J_d^X = k^* g_* \left( \frac{\alpha_F^*[Q_d^X]/g^* c_{\text{top}}(E)}{[F/Q_d^Y]} \right)$$

and if (as will be our case) the right side is well-defined as a  $\mathbb{C}^*$ -equivariant cohomology class, then the equality holds in  $\mathbb{C}^*$ -equivariant cohomology, taking limits.

**Remark:** The lemma is an easy exercise using the Atiyah-Bott localization theorem. It immediately implies the quantum Lefschetz theorem for quadrics, taking  $Y = \mathbb{P}^n$  and  $Q_d^Y = \mathbb{P}_d^n$ , with  $[Q_d^X] = [X_d]$  defined by the  $2d+1$  induced quadratic equations. Indeed, the obstruction to an easy proof of the quantum Lefschetz theorem for Fano complete intersections  $X \subset \mathbb{P}^n$  is simply the fact that the linear sigma models  $X_d \subset \mathbb{P}_d^n$  tend to have “extra” components aside from the closure of  $\text{Map}_d(\mathbb{P}^1, X)$ , making property  $(\dagger)$  difficult to check. Such extra components do not exist when  $X$  is a homogeneous space.

**(6.2) Applying the Lemma.** Let

$$Q_d^{\mathbb{G}(s,n)}$$

be the Grothendieck quot scheme of vector bundle subsheaves  $K \subset \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^1}$  of degree  $-d$  and rank  $s$  on  $\mathbb{P}^1$ . This is an alternative smooth compactification of the Hilbert space  $\text{Map}_d(\mathbb{P}^1, \mathbb{G}(s,n))$  with universal sub and quotient sheaves:

$$0 \rightarrow \mathcal{K} \rightarrow \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^1 \times Q_d^{\mathbb{G}}} \rightarrow \mathcal{Q} \rightarrow 0$$

As we showed in our paper [BCK], under the natural map to the linear sigma model:

$$v : \begin{array}{ccc} Q_d^{\mathbb{G}(s,n)} & \rightarrow & Q_d^{\mathbb{G}(1, \binom{n}{s})} = \mathbb{P}_d^{\binom{n}{s}-1} \\ \cup & & \cup \\ F & \xrightarrow{g} & \mathbb{G}(s,n) \end{array}$$

the components  $F_{\vec{d}} \subset F$  are in bijection with the possible splitting types:

$$K = \bigoplus_{i=1}^s \mathcal{O}_{\mathbb{P}^1}(-d_i)$$

Associated to each *splitting type*  $\vec{d} = (d_1, d_2, \dots, d_s)$  (in non-decreasing order, with  $d_1 = \dots = d_{s_1}, d_{s_1+1} = \dots = d_{s_2}, \dots$ ) is  $F_{\vec{d}} \cong Fl(s_1, \dots, s_l = s, n)$ . The Euler classes to the  $F_{\vec{d}}$  were inverted and pushed forward in [BCK]. This required a lemma of Brion ([Bri], see Lemma 1.4 of [BCK]) and the following implication:

(6.2.1) Let  $P(x_1, \dots, x_s, d_1, \dots, d_s)$  be a polynomial with the property that:

$$\prod_{1 \leq i \leq l} \prod_{1 \leq a < b \leq s_i - s_{i-1}} (x_{s_{i-1}+b} - x_{s_{i-1}+a}) \text{ divides } P(x_1, \dots, x_s, \vec{d})$$

for each splitting type  $\vec{d}$  and such that each:

$$\frac{P(H_1, \dots, H_s, \vec{d})}{\prod_i \prod_{a < b} (H_{s_{i-1}+b} - H_{s_{i-1}+a})}$$

represents a cohomology class in  $F_{\vec{d}}$ , where the  $H_1, \dots, H_s$  are the Chern roots of  $S$ , arranged so that  $H_{s_{i-1}+1}, \dots, H_{s_i}$  are the Chern roots of  $S_i/S_{i-1}$  on  $F_{\vec{d}}$ . Then:

$$\sum_{\vec{d}} g_* \left( \frac{P(H_1, \dots, H_s, \vec{d})}{\prod_i \prod_{a < b} (H_{s_{i-1}+b} - H_{s_{i-1}+a})} \right) = \sum_{d_1 + \dots + d_s = d} \frac{P(H_1, \dots, H_s, d_1, \dots, d_s)}{\prod_{1 \leq j < j' \leq s} (H_{j'} - H_j)}$$

**Clarification:** The polynomial  $P(x_1, \dots, x_r)$  in Lemma 1.4 of [BCK] corresponds here to:

$$\frac{P(x_1, \dots, x_s, \vec{d})}{\prod_i \prod_{a < b} (x_{s_{i-1}+b} - x_{s_{i-1}+a})}$$

As we showed in [BCK]:

$$\frac{1}{[F_{\vec{d}}/Q_d^{\mathbb{G}}]} = \frac{P_{\hbar}(H_1, \dots, H_s, \vec{d})}{\prod_i \prod_{a < b} (H_{s_{i-1}+b} - H_{s_{i-1}+a})}$$

with

$$P_{\hbar}(x_1, \dots, x_s, d_1, \dots, d_s) = (-1)^{(s-1)(\sum d_i)} \frac{\prod_{1 \leq j < j' \leq s} (x_{j'} - x_j + (d_{j'} - d_j)\hbar)}{\prod_{i=1}^s \prod_{k=1}^{d_i} (x_i + k\hbar)^n}$$

Our point here is that (6.2.1) also applies to classes:

$$\frac{\alpha_{F_{\vec{d}}}^*[Q_d^X]/c_{\text{top}}(E)}{[F_{\vec{d}}/Q_d^{\mathbb{G}}]}$$

(from(6.1.1)) describing the  $J$ -functions of sub-homogeneous spaces. For example, consider the ‘‘Lagrangian Grassmannian:’’

$$L\mathbb{G} \subset \mathbb{G}(n, 2n) \subset \mathbb{P}^{\binom{2n}{n}-1}$$

obtained as the zero scheme of the section of  $E = \wedge^2(S^\vee)$  over  $\mathbb{G}(n, 2n)$  induced from the standard algebraic symplectic form on  $\mathbb{C}^{2n}$ .



**Theorem 2.** The  $J$ -function of the Lagrangian Grassmannian is given by:

$$J_d^{LG} = \sum_{d_1 + \dots + d_n = d} \left( \frac{\prod_{n \geq i > j \geq 1} \prod_{k=0}^{d_i + d_j} (H_i + H_j + k\hbar)}{\prod_{n \geq i > j \geq 1} (H_i + H_j)} \right) \cdot \left( \frac{(-1)^{(n-1)d}}{\prod_{i=1}^n \prod_{k=1}^{d_i} (H_i + k\hbar)^{2n}} \cdot \prod_{n \geq i > j \geq 1} \frac{(H_i - H_j + (d_i - d_j)\hbar)}{(H_i - H_j)} \right)$$

**Proof:** Consider the corresponding section of the vector bundle:

$$\pi_* \wedge^2 (\mathcal{K}^\vee) \text{ on } Q_d^{\mathbb{G}(n, 2n)}$$

and its zero scheme, the ‘‘Lagrangian Quot scheme’’  $LQ_d^{\mathbb{G}} \subset Q_d^{\mathbb{G}(n, 2n)}$  (here  $\pi$  is the projection  $\mathbb{P}^1 \times Q_d^{\mathbb{G}(n, 2n)} \rightarrow \mathbb{P}^1$ ). We apply (6.1.1) with:

$$[Q_d^{LG}] = c_{\text{top}}(\pi_*(\wedge^2 \mathcal{K}^\vee))$$

noticing that  $v_*[Q_d^{LG}] = u_*[G_d^{LG}]$  by virtue of the fact that  $LG$  is homogeneous, and the zero section of  $\pi_* \wedge^2 \mathcal{K}^\vee$  is transverse along the Hilbert scheme of maps  $\text{Map}_d(\mathbb{P}^1, LG)$ . So all that remains is the computation of  $c_{\text{top}}(\pi_* \wedge^2 \mathcal{K}^\vee)$  restricted to the fixed loci  $F_{\vec{d}}$ . But this is easily done.  $F_{\vec{d}}$  consists of vector bundle subsheaves of splitting type  $\vec{d}$ , and there is a filtration of the restriction of  $\mathcal{K}$  to  $\mathbb{P}^1 \times F_{\vec{d}}$ :

$$\mathcal{K}_1 \subset \mathcal{K}_2 \subset \dots \subset \mathcal{K}_l = \mathcal{K} \subset \mathcal{O}_{\mathbb{P}^1 \times F_{\vec{d}}}$$

with  $\mathcal{K}_i/\mathcal{K}_{i-1} \cong \pi^*(S_i/S_{i-1})(-d_i D_0)$  where  $D_0 = 0 \times F_{\vec{d}}$ . From this, one computes (as in [BCK], in the computations preceding Lemma 1.4):

$$c_{\text{top}}(\pi_* \wedge^2 \mathcal{K}^\vee)|_{F_{\vec{d}}} = Q_{\hbar}(H_1, \dots, H_n, \vec{d})$$

where

$$Q_{\hbar}(x_1, \dots, x_n, d_1, \dots, d_n) = \prod_{i > j} \prod_{k=0}^{d_i + d_j} (x_i + x_j + k\hbar)$$

and then applying (6.2.1) to  $P_{\hbar} \cdot \left( Q_{\hbar} / \prod_{i > j} (x_i + x_j) \right)$  gives the Theorem, since  $c_{\text{top}}(E) = \prod_{i > j} (x_i + x_j)$ . Note that evidently  $\prod_{i > j} (x_i + x_j)$  divides  $Q_{\hbar}$ , hence the Theorem is valid in  $\mathbb{C}^*$ -equivariant cohomology.

We next prove Theorem 1 in the same way, considering:

$$Fl(s_1, \dots, s_l, n) \subset \prod_{i=1}^l \mathbb{G}(s_i, n) \subset \mathbb{P}^N$$

with  $E = \bigoplus_{i=1}^{l-1} \mathcal{H}om(S_i, Q_{i+1}) \cong \bigoplus_{i=1}^{l-1} S_i^\vee \otimes Q_{i+1}$  (and  $\prod \mathbb{G} \subset \mathbb{P}$  embedded by Plücker and Segre). Here the zero scheme of the section of the bundle on the

product of Quot schemes:

$$\pi_* \left( \bigoplus_{i=1}^{l-1} \mathcal{K}_i^\vee \otimes V/\mathcal{K}_{i+1} \right)$$

is smooth and irreducible ( $V$  is the trivial rank  $n$  bundle). We will denote the class of the zero section by  $[Q_{d_i}^{HG}]$ . It is the fundamental class of the “hyperquot” scheme of flags of vector bundle subsheaves of the trivial bundle on  $\mathbb{P}^1$ :

$$K_1 \subset \dots \subset K_l \subset \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^1}$$

Within the product of Quot schemes, the fixed components relevant to our computations are the products of flag manifolds  $\prod_{i=1}^l F_{\vec{d}_i}$  indexed by multi-splitting types: (Note: It is much harder to describe the fixed components of the hyperquot scheme. See [LLY]). Now we proceed as in the proof of Theorem 2. In this case, by (6.1.1) we need to compute the push-forward:

$$g_* \left( \sum_{\vec{d}_i} \alpha_{\prod F_{\vec{d}_i}}^* [Q_{d_i}^{HG}] / c_{\text{top}}(E) \right)$$

to the product of Grassmannians. This follows from an obvious generalization of (6.2.1) to the product of Grassmannians. When applied to:

$$\prod P_{\vec{h}}(x_{i,1}, \dots, x_{i,s_i}, d_{i,1}, \dots, d_{i,s_i})$$

we obtain the  $J$ -function for the product of Grassmannians. We need only, therefore, to give the analogue of the polynomial  $Q_{\vec{h}}$  from the proof of Theorem 2. For this, we notice that the sheaf  $\mathcal{K}_i^\vee \otimes V/\mathcal{K}_{i+1}$  is not a vector bundle (though its push-forward is a vector bundle). For this reason, it is more reasonable to work with the top Chern class via the exact sequence:

$$0 \rightarrow \mathcal{K}_i^\vee \otimes \mathcal{K}_{i+1} \rightarrow \mathcal{K}_i^\vee \otimes V \rightarrow \mathcal{K}_i^\vee \otimes V/\mathcal{K}_{i+1} \rightarrow 0$$

and its pushforward to the product of quot schemes:

$$\begin{aligned} 0 \rightarrow \pi_*(\mathcal{K}_i^\vee \otimes \mathcal{K}_{i+1}) \rightarrow \pi_*(\mathcal{K}_i^\vee \otimes V) \rightarrow \pi_*(\mathcal{K}_i^\vee \otimes V/\mathcal{K}_{i+1}) \rightarrow \\ \rightarrow R^1 \pi_*(\mathcal{K}_i^\vee \otimes \mathcal{K}_{i+1}) \rightarrow 0 \end{aligned}$$

When the top Chern class is computed with this sequence, we obtain our desired polynomial  $Q_{\vec{h}}(x_{i,j}, d_{i,j})$  which is divisible by  $c_{\text{top}}(E)$  and the quotient precisely evaluates to (5.2.2) when we set  $x_{i,j} = H_{i,j}$ . This proves Theorem 1.

Finally, the reader may immediately generalize Theorems 1 and 2 to give the  $J$ -functions for all isotropic flag manifolds, i.e., the homogeneous spaces  $G/P$  for  $G = SO_{2n+1}(\mathbb{C}), Sp_{2n}(\mathbb{C}), SO_{2n}(\mathbb{C})$  by realizing the isotropic flag manifold inside the appropriate product of Grassmannians as the zero locus of a section of an appropriate homogeneous vector bundle. The point here is

that a flag of vector subspaces is isotropic for a symmetric or symplectic form iff the largest subspace in the flag is. This case is therefore a combination of the ones treated in Theorems 1 and 2: the isotropic flag manifold is the zero locus of the natural section of a direct sum of  $Hom$  bundles as in Theorem 1, and either the second symmetric power (in the symmetric case), or the second exterior power (in the symplectic case) of the dual to the tautological bundle on the appropriate factor.

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