Quasimap Theory

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Abstract. We provide a short introduction to the theory of ε-stable quasimaps and its applications via wall-crossing to Gromov-Witten theory of GIT targets.

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1. Introduction

This note is intended as a brief survey of the theory of quasimaps from curves to a certain (large) class of GIT quotients, and of its applications to Gromov-Witten theory, as developed in the papers [10, 14, 11, 12, 13, 6]. The theory may be viewed as an algebro-geometric realization of Witten's Gauged Linear σ-model (GLSM) [52] in the geometric phases. The study of GLSM and of its relation to Mirror Symmetry has been a very active area in String Theory, see [42, 32, 31, 22, 1, 34] for a (very incomplete) sampling of developments.

When such a geometric phase (a target with a GIT presentation) is fixed, there is a family of quasimap theories indexed by a stability parameter $\varepsilon \in \mathbb{Q}_{>0}$. When $\varepsilon > 1$ one recovers the "nonlinear σ-model", i.e., the Gromov-Witten theory of the target. There is a wall-and-chamber structure on $\mathbb{Q}_{>0}$, with walls at $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{d}, \dots$, such that the theory stays unchanged in the chamber $(\frac{1}{d+1}, \frac{1}{d}]$. Wall-crossing formulas relating the invariants in different chambers of nonsingular targets are conjectured (and are established in many cases) in [11] for genus zero, and in [12] for all genera; the genus zero case is extend to orbifolds in [6]. These results are described in §4-5 of the paper. As explained there, the wall-crossing formulas may be viewed as generalizations in many directions of Givental's Mirror Theorems [26] for (complete intersections in) toric manifolds with semi-positive anti-canonical class. In addition, the *mirror map* is given a geometric interpretation as the generating series of primary quasimap invariants with a fundamental class insertion.

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There is also an extension of the theory in a different direction, allowing the domain curves of quasimaps to carry weighted markings. When (some) markings are given infinitesimally small weights, this produces for many targets a *closed form* expression of a "big I-function" defined on the entire parameter space $H^*(X,\mathbb{Q})$ associated to the GIT target X. By a result in [13] the big J-function of the Gromov-Witten theory of X is obtained from this new big I-function via the "Birkhoff factorization" procedure of [18]. As a result, one obtains an explicit determination of all the genus zero Gromov-Witten invariants of X.

2. Maps from curves to quotient targets

2.1. A class of GIT quotients. Let $W = \mathrm{Spec} A$ be an affine algebraic variety over $\mathbb{C}$ and let $\mathbf{G}$ be a reductive algebraic group over $\mathbb{C}$, acting upon W from the right. Choose a character of $\mathbf{G}$, $\theta \in \mathrm{Hom}(\mathbf{G}, \mathbb{C}^*)$. Denote by $\mathbb{C}_\theta$ the associated 1-dimensional $\mathbf{G}$-representation space. This determines a $\mathbf{G}$-equivariant line bundle $L_\theta := W \times \mathbb{C}_\theta$ on W.

There are four quotients as follows.

(i) The *affine quotient* $W/_{\mathrm{aff}}\mathbf{G} := \mathrm{Spec}(A^{\mathbf{G}})$, which is of finite type over $\mathbb{C}$ by Hilbert's Theorem.

(ii) The *stack quotient* $[W/\mathbf{G}]$ (see [20, 40]). One departure when working with algebraic stacks versus working with schemes is that algebraic stacks are groupoid-valued functors from the category of schemes, while schemes are set-valued functors. By the Yoneda lemma, the category $(\mathrm{Sch}/\mathbb{C})$ of $\mathbb{C}$-schemes is embedded into the category of functors of points. For schemes $X, Y \in (\mathrm{Sch}/\mathbb{C})$, the set of Y-points of X is $\mathrm{Hom}_{(\mathrm{Sch}/\mathbb{C})}(Y, X)$, i.e., the set of all morphisms from Y to X over $\mathbb{C}$. The stack $[W/\mathbf{G}]$ can be considered as a functor from $(\mathrm{Sch}/\mathbb{C})$ to the category of groupoids, defined as follows. A morphism from Y to $[W/\mathbf{G}]$ is by definition a triple $(Y, P, \tilde{f})$, where P is a principal $\mathbf{G}$-bundle on Y (which is trivializable in the étale topology of Y) and $\tilde{f} : P \to W$ is a $\mathbf{G}$-equivariant morphism. Equivalently, it is a triple (Y, P, f), with f a section of the induced fiber bundle $P \times_{\mathbf{G}} W \to Y$ with fiber W. An isomorphism from (Y, P, f) to (Y, P', f') is a $\mathbf{G}$-bundle homomorphism $\varphi : P \to P'$ such that $f' \circ \varphi = f$. Suppose that $Y = \mathrm{Spec}\mathbb{C}$; then the $\mathbb{C}$-points of $[W/\mathbf{G}]$ form a groupoid, the collection of orbits with the isomorphisms described above. A $\mathbb{C}$-point has non-trivial automorphisms if and only if the corresponding $\mathbf{G}$-orbit in Y has non-trivial stabilizer group.

Consider the trivial $\mathbf{G}$-bundle $W \times \mathbf{G}$ on W. It comes with the $\mathbf{G}$-equivariant map $W \times \mathbf{G} \to W$ given by the action. This gives a canonical morphism from W to $[W/\mathbf{G}]$, fitting in the cartesian diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & W \\ \downarrow & & \downarrow \\ Y & \longrightarrow & [W/\mathbf{G}]. \end{array}$$

The geometry of $[W/\mathbf{G}]$ is encoded by the "atlas" $W \to [W/\mathbf{G}]$.

(iii) The *GIT quotient* $W/\!\!/\mathbf{G} := W/\!\!/_\theta\mathbf{G} := \mathrm{Proj}(\oplus_{n\geq 0}\Gamma(W, L_\theta^{\otimes n})^{\mathbf{G}})$. This is a quasi-projective scheme, equipped with a canonical *projective* morphism to $W/_{\mathrm{aff}}\mathbf{G}$. It is called the *Proj quotient in direction* θ in [43, §6.13].

(iv) The *GIT stack quotient* $[W^{ss}/\mathbf{G}]$. This is an open substack of $[W/\mathbf{G}]$ since $W^{ss} := \{p \in W : s(p) \neq 0 \text{ for some } n > 0, s \in \Gamma(W, L_\theta^n)^{\mathbf{G}}\}$ is a $\mathbf{G}$-invariant open subset of W.

Assumption 1.1: $\mathbf{G}$ acts on W^{ss} with at most finite stabilizers.

Assumption 1.1 is equivalent to requiring that there are no strictly semi-stable points. Under this assumption, the GIT stack quotient $[W^{ss}/\mathbf{G}]$ is a Deligne-Mumford (DM) stack.

It follows that there is a natural commuting diagram of morphisms:

$$\begin{array}{ccc} [W^{ss}/\mathbf{G}] & \longrightarrow & [W/\mathbf{G}] \\ \text{proper}\downarrow & & \downarrow \\ W/\!\!/\mathbf{G} & \xrightarrow[\text{projective}]{} & W/_{\mathrm{aff}}\mathbf{G}. \end{array}$$

The left vertical morphism is proper, see e.g. [35].

2.2. Examples.

1. *Projective Spaces.* Let $\mathbf{G} = \mathbb{C}^*$ diagonally act on $V = \mathbb{C}^{n+1}$ and let $\theta = \mathrm{id}_{\mathbb{C}^*}$. Then
$$V/\!\!/\mathbf{G} = \mathbb{P}^n \subset [V/\mathbf{G}] = [\mathbb{C}^{n+1}/\mathbb{C}^*]$$
and L_θ restricted to $V/\!\!/\mathbf{G}$ is $\mathcal{O}(1)$.

   Note that the set of $\mathbb{C}$-points of $[V/G]$ contains one more element $[0, ..., 0]$ other than those in the projective space $\mathbb{P}^n$. This point has nontrivial automorphisms and is called a stacky point. Even when $n = 0$, the stack $[\mathbb{C}/\mathbb{C}^*]$ is interesting. This stack parameterizes pairs (L, s) with L a line bundle and s a section of L.

2. *Grassmannians.* Let $V = \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$, $\mathbf{G} = GL(r, \mathbb{C})$ and $\theta = \det$. Then $V/\!\!/\mathbf{G} = Gr(r, n)$, the Grassmannian of r-planes in $\mathbb{C}^n$. A similar description works for a type A flag variety, see e.g. [5].

3. *Toric DM-stacks.* $V = \mathbb{C}^N$ and $\mathbf{G} = (\mathbb{C}^*)^r$; there are many choices of θ and the GIT quotient $[V^{ss}/\mathbf{G}]$ is a toric DM-stack.

4. *Complete Intersections.* Any projective variety $X \subset \mathbb{P}^{n-1}$ is a GIT quotient: $X = W/\!\!/\mathbb{C}^*$, with $W = C(X) \subset \mathbb{C}^n$, the affine cone over X, but only *complete intersections* lead to good theories (see Remark 3.3 below).

5. *Zero locus of regular sections of homogenous vector bundles.* Let $V, \mathbf{G}, \theta$ define a GIT quotient as in §2.1 and let E be a $\mathbf{G}$-representation with induced vector bundle $\mathcal{E} = V^{ss} \times_{\mathbf{G}} E$ on $V/\!\!/\mathbf{G}$. Let $s \in \Gamma(V, V \times E)^{\mathbf{G}}$ be regular with induced $\bar{s} \in \Gamma(V/\!\!/\mathbf{G}, \mathcal{E})$. If we set $W := Z(s) \subset V$ (note that W is lci), then $W/\!\!/\mathbf{G} = Z(\bar{s}) \subset V/\!\!/\mathbf{G}$. For example, complete intersections in toric varieties are obtained in this way, but there are many more non-abelian examples with indecomposable bundles $\mathcal{E}$ which are *not* complete intersection.

   According to Coates, Corti, Galkin, and Kasprzyk [17] who rework the Mori-Mukai classification of Fano 3-folds, every smooth Fano 3-fold can be realized as an example of this type. We remark that the Rødland's Pfaffian Calabi-Yau 3-fold and the determinantal Gulliksen-Negård Calabi-Yau 3-fold are also of this type (see [50, §2], [34, §5] respectively).

6. *Nakajima Quiver Varieties.* Nakajima quiver varieties ([47]) give a large class of typically quasi-projective only GIT quotients of the kind we are interested in, see [14, Example 6.3.2]. Particularly interesting such examples are certain Hilbert schemes of points on non-compact surfaces. For example, let

$$\begin{aligned} V &= \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)^{\oplus 2} \oplus \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}), \\ W &:= \{(A, B, i, j) \in V \mid [A, B] + ij = 0\}, \\ \mathbf{G} &= GL(n, \mathbb{C}), \text{ and } \theta = \det. \end{aligned}$$

   Then $W/\!\!/\mathbf{G} = \mathrm{Hilb}_n(\mathbb{C}^2)$ and $W/_{\mathrm{aff}}\mathbf{G} = \mathrm{Sym}^n(\mathbb{C}^2)$. This is the well-known ADHM presentation of the Hilbert scheme of points in the plane.

   More generally, let $\Gamma \subset SL(2, \mathbb{C})$ be a finite subgroup. Let

$$X := \Gamma\text{-Hilb}(\mathbb{C}^2) := \{Z \subset \mathbb{C}^2 : \mathcal{O}_Z \cong \mathbb{C} \cdot \Gamma\}.$$

   It is the crepant resolution of $\mathbb{C}^2/\Gamma$. Using an appropriate Fourier-Mukai functor $\Phi : D(X) \to D^\Gamma(\mathbb{C}^2)$, the Hilbert scheme $\mathrm{Hilb}_n(X)$ can be realized as the Nakajima quiver variety associated to the framed affine Dynkin diagram with a certain King's stability condition, see [37], [48].

7. *Local Targets.* Let $V, \mathbf{G}, \theta$ define a projective GIT quotient and let E be a $\mathbf{G}$-representation space, with an induced vector bundle $\mathcal{E} = V^s \times_{\mathbf{G}} E$ on $V/\!\!/\mathbf{G}$. Assume E is a sum of $\mathbb{C}_{k_i\theta}$ for some negative integers $k_1, ..., k_r$. If $W := V \times E$, then θ gives a linearization and $W/\!\!/\mathbf{G}$ is the total space of $\mathcal{E}$ over $V/\!\!/\mathbf{G}$. Again, it is only quasi-projective. These are usually called *local targets* in Gromov-Witten theory.

8. $SU_C(2, L)$. Let C be a nonsingular projective curve. Then the moduli space of rank 2 stable vector bundles on C with an odd determinant L, $\deg L \geq 4g(C) - 1$ is realized as the GIT quotient of an affine variety by a general linear group (see [43, Theorem 10.1]).

2.3. Moduli of maps to the stack quotient.

To keep the presentation simple, from now on we assume that *the* $\mathbf{G}$*-action on* W^{ss} *is free.* The general case is referred to [6].

Let $(C, p_1, \cdots, p_k)$ be a pointed, genus g prestable curve, i.e., C is a connected projective curve at worst with nodal singularities, p_i are ordered nonsingular points of C, and the arithmetic genus of C is g.

As explained, a map

$$C \xrightarrow{[u]} [W/\mathbf{G}]$$

is described by the data

$$\big((C, \{p_i\}), P, u\big).$$

with P a principal $\mathbf{G}$-bundle on C and u a section of the induced W-bundle $P \times_{\mathbf{G}} W \xrightarrow{\rho} C$. Any such $[u] : C \to [W/\mathbf{G}]$ defines

$$\beta \in \mathrm{Pic}([W/\mathbf{G}])^{\vee} = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}^{\mathbf{G}}(W), \mathbb{Z}), \quad \beta(L) = \deg_C \mathcal{L},$$

where $\mathcal{L} := u^*(P \times_{\mathbf{G}} L)$ (a line bundle on C). This β is called the *numerical class* of the triple $\big((C, \{p_i\}), P, u\big)$.

Consider the moduli stack $\mathfrak{M}_{g,k}([W/\mathbf{G}], \beta)$ parametrizing all $\big((C, \{p_i\}), P, u\big)$ as above. It is a non-separated Artin stack of infinite type. We describe here its obstruction theory. Consider the natural morphisms:

$$\begin{array}{ccc} \mathfrak{M}_{g,k}([W/\mathbf{G}], \beta) & \xrightarrow{\mu} & \mathfrak{B}un_{\mathbf{G}}^{g,k} \\ & \searrow^{\nu} & \downarrow{\scriptstyle\phi} \\ & & \mathfrak{M}_{g,k} \end{array}$$

where

- $\mathfrak{M}_{g,k}$ is the moduli stack of prestable k-pointed curves of genus g;
- $\phi : \mathfrak{B}un_{\mathbf{G}}^{g,k} \longrightarrow \mathfrak{M}_{g,k}$ is the relative moduli stack of principal $\mathbf{G}$-bundles on the fibers of the universal curve $\mathfrak{C}_{g,k} \longrightarrow \mathfrak{M}_{g,k}$;
- μ and ν are the natural forgetful morphisms.

Both $\mathfrak{M}_{g,k}$ and $\mathfrak{B}un_{\mathbf{G}}^{g,k}$ are smooth Artin stacks and ϕ is a smooth morphism. It follows that the natural obstruction theory to consider is the μ-relative obstruction theory governing deformations of sections u. Over the open substack $M_{g,k}(W/\!\!/\mathbf{G}, \beta)$, this induces the usual *absolute* obstruction theory of maps to the GIT quotient. The stack $M_{g,k}(W/\!\!/\mathbf{G}, \beta)$ parameterizes the triples $\big((C, \{p_i\}), P, u\big)$ with C irreducible and the image of u contained in $P \times_{\mathbf{G}} W^{ss}$.

The above discussion suggests several natural questions to address:

1. The Kontsevich compactification $\overline{M}_{g,k}(W/\!\!/\mathbf{G},\beta)$ is an open and closed substack of $\mathfrak{M}_{g,k}([W/\mathbf{G}],\beta)$. Using the linearization θ, can we impose *stability conditions* to single out other *Deligne-Mumford* open and closed substacks containing $M_{g,k}(W/\!\!/\mathbf{G},\beta)$, and which are *proper* (over $W/_{\mathrm{aff}}\mathbf{G}$)?

2. If in addition the restriction of obstruction theory is perfect, these substacks will have a virtual class, hence we get "numerical invariants" associated to the triple $(W,\mathbf{G},\theta)$. When is the obstruction theory perfect?

3. Assuming the first two questions have been answered satisfactorily, how do the invariants change when varying the stability condition? Can one obtain explicit "wall-crossing" formulas?

In the rest of the paper we explain how quasimap theory provides some answers to the above questions. The first two questions are discussed in §3, while §4 and §5 deal with the wall-crossing phenomenon and its relation to Mirror Symmetry.

3. Quasimaps and ε-stability

3.1. Stable quasimaps.

Definition 3.1. (i) $\big((C,\{p_i\}),P,u\big)$ is called a θ-**quasimap** (or simply quasimap) to $W/\!\!/\mathbf{G}$ if

$$\#\{u(C)\cap W^{us}\}<\infty,$$

where $W^{us}:=W\setminus W^{ss}$. Hence, $C \overset{[u]}{\dashrightarrow} W/\!\!/\mathbf{G}$ is a rational map with finitely many base points.

(ii) A θ-quasimap is called **prestable**, if the base points are away from the nodes and markings of C.

For such a prestable quasimap and $x\in C$, define

$$\ell(x):=\mathrm{length}(\mathcal{O}_{x,C}/[u]^\sharp I_{[W^{us}/\mathbf{G}]}\mathcal{O}_{x,C})\in\mathbb{Z}_{\geq 0}.$$

(iii) Fix $\varepsilon\in\mathbb{Q}_{>0}$. A prestable quasimap is called ε-**stable**, if

1. the line bundle $\omega_C(\sum p_i)\otimes\mathcal{L}_\theta^\varepsilon$ on C is ample, where $\mathcal{L}_\theta=u^*(P\times_{\mathbf{G}}L_\theta)=P\times_{\mathbf{G}}\mathbb{C}_\theta$.
2. $\varepsilon\ell(x)\leq 1$ for every nonsingular point $x\in C$.

There is also an "asymptotic" stability condition, obtained by requiring only the ampleness condition, but for *every* $\varepsilon\in\mathbb{Q}_{>0}$. We denote it by $\varepsilon=0+$.

Denote by $\mathrm{Q}^{\varepsilon}_{g,k}(W/\!\!/\mathbf{G},\beta)$ the moduli stack parameterizing ε-stable quasimaps of type (g,k,β).

Theorem 3.2. ([14]) *For every* $\varepsilon \geq 0+$, $\mathrm{Q}^{\varepsilon}_{g,k}(W/\!\!/\mathbf{G}, \beta)$ *is a DM stack with a natural proper morphism to the affine quotient. (In particular, if* $W/\!\!/\mathbf{G}$ *is projective, then* $\mathrm{Q}^{\varepsilon}_{g,k}(W/\!\!/\mathbf{G}, \beta)$ *is proper.)*

If W^{ss} *is nonsingular and* W *has at worst lci singularities (necessarily in* W^{us}*), then the canonical obstruction theory on* $\mathrm{Q}^{\varepsilon}_{g,k}(W/\!\!/\mathbf{G}, \beta)$ *(relative to* $\mathfrak{B}un^{g,k}_{\mathbf{G}}$*) is perfect.*

From now on we assume the lci condition on W, so that $\mathrm{Q}^{\varepsilon}_{g,k}(W/\!\!/\mathbf{G}, \beta)$ carries a virtual fundamental class.

Remark 3.3. 1. The theory depends on the triple $(W, \mathbf{G}, \theta)$, not just on the geometric target $W/\!\!/\mathbf{G}$.

2. Assume $(g, k) \neq (0, 0)$.

   - If $\varepsilon > 1$ we get the Kontsevich stable maps to $W/\!\!/\mathbf{G}$; the obstruction theory is then perfect for *all* W (of course W^{ss} is assumed to be nonsingular). However, for $\varepsilon \leq 1$ the lci condition is necessary.
   - If $0 < \varepsilon \leq \frac{1}{\beta(L_\theta)}$, all lengths of base points are allowed and the domain curve has *no rational tails*. The asymptotic stability condition says that we are in this chamber for *all* β.
   - There are finitely many "chambers" $(\frac{1}{d+1}, \frac{1}{d}]$ such that the moduli spaces stay constant for $\varepsilon \in (\frac{1}{d+1}, \frac{1}{d}]$; intuitively, when crossing the wall we trade rational tails of degree d (with respect to $\mathcal{O}(\theta)$) with base points of length d.

3.2. Some history.

- For *fixed* curve with *no markings* and $\varepsilon = 0+$, many earlier compactifications are unified by this construction:
  - Drinfeld's quasimaps to $\mathbb{P}^n$, see [3]. However, note that the moduli of Drinfeld's quasimaps to flag varieties considered in [3] are defined using the Plücker embeddings and therefore fit into the situation described in Example 4 of §2.2. Since under the Plücker embedding the flag varieties are not complete intersections, the canonical obstruction theory of the moduli spaces is not perfect.
  - Gauged linear σ-models for toric targets (Witten [52], Morrison - Plesser [42], Givental [25, 26]).
  - Quot schemes for Grassmannians ((Strømme [49], Bertram [2]); their generalization to type A flag varieties due to Laumon [38, 39] (and rediscovered under the names hyperquot or flag-quot schemes in [9, 36]).
  - ADHM sheaves for $\mathrm{Hilb}_n(\mathbb{C}^2)$ (Diaconescu [21]).
- For the case when the complex structure of the domain curves varies and/or markings are allowed, the starting point was the work by Marian, Oprea,

and Pandharipande [41] on moduli of stable quotients (in the terminology introduced above, this corresponds to target a Grassmannian and $\varepsilon = 0+$). Inspired by their paper, the authors developed the toric case and realized that the GIT point of view is the correct generalization of both ([10]). The ε-stability idea appeared first in work by Mustaţă and Mustaţă for target $\mathbb{P}^n$ ([46]). For Grassmannian targets, Toda introduced and studied ε-stable quotients in [51].

- There's a long (ongoing) related story in the symplectic category concerned with the study of (compactifications of) the moduli space of solutions to vortex equations, starting with work of Cieliebak - Gaio - Salamon and of Mundet i Riera, see [7, 8, 44, 45, 29, 30, 54]. An algebraic version of this theory is developed by Woodward in [53].
- Frenkel - Teleman - Tolland are developing a general formalism of a Gromov-Witten type theory of quotient stacks $[Y/G]$, see [24].

3.3. Quasimap Invariants. There are evaluation maps ev_i to $W/\!\!/\mathbf{G}$ (by the prestable condition) and tautological line bundles M_i on $\mathrm{Q}^{\varepsilon}_{g,k}(W/\!\!/\mathbf{G},\beta)$ with fiber the cotangent line to C at the i^{th} marking:

$$ev_i : \mathrm{Q}^{\varepsilon}_{g,k}(W/\!\!/\mathbf{G},\beta) \to W/\!\!/\mathbf{G}.$$

As usual, denote $\psi_i := c_1(M_i)$. Given

$$\delta_1, \dots \delta_k \in H^*(W/\!\!/\mathbf{G}, \mathbb{Q})$$

and integers $a_1, \dots, a_k \geq 0$, we define ε-quasimap invariants

$$\langle \delta_1\psi_1^{a_1}, \dots, \delta_k\psi_k^{a_k}\rangle^{\varepsilon}_{g,k,\beta} := \int_{[\mathrm{Q}^{\varepsilon}_{g,k}(W/\!\!/\mathbf{G},\beta)]^{\mathrm{vir}}} \prod \psi_i^{a_i} \prod ev_i^*(\delta_i)$$

for all $\varepsilon \geq 0+$.

If $\varepsilon > 1$ (we write $\varepsilon = \infty$ for all such stability conditions), these are the descendant Gromov-Witten invariants of $W/\!\!/\mathbf{G}$.

The definition above requires $W/\!\!/\mathbf{G}$ projective; in the quasi-projective case there are equivariant versions available for all interesting targets, e.g., toric varieties, local targets, and Nakajima quiver varieties. Precisely, what is needed in order to have a good theory for non-compact targets $W/\!\!/\mathbf{G}$ is that there is an action on W by an algebraic torus $\mathbf{T} \cong (\mathbb{C}^*)^r$, commuting with the $\mathbf{G}$-action and such that the $\mathbf{T}$-fixed locus on the affine quotient $W/_{\mathrm{aff}}\mathbf{G}$ is proper (and therefore a finite set). To get a unified framework, we will make this assumption from now, allowing the case $r = 0$ of a trivial torus.

The invariants satisfy the "splitting axiom" and in fact they form the degree zero sector of a Cohomological Field Theory (CohFT) on $H^*(W/\!\!/\mathbf{G})$. However, for general targets $W/\!\!/\mathbf{G}$ and $\varepsilon \leq 1$, the *string equation* may fail so the CohFT will not have a flat identity. We refer the reader to [12, §2] for some more details on the quasimap CohFT.

4. Genus zero wall-crossing and mirror maps

It is natural to expect that different stability chambers carry the same information. This will be expressed via wall-crossing formulas for generating functions of the invariants. In this section we explain why the wall-crossing formulas for genus zero invariants are generalizations (in many directions) of Givental's Toric Mirror Theorems.

First we fix some notations:

- $H^*(W/\!\!/\mathbf{G})$ denotes the localized $\mathbf{T}$-equivariant cohomology with $\mathbb{Q}$-coefficients.
- $\langle\ ,\ \rangle$ is the intersection pairing on $H^*(W/\!\!/\mathbf{G})$.
- $\{\gamma_1 = \mathbb{1}, \ldots, \gamma_s\}$ and $\{\gamma^1, \ldots, \gamma^s\}$ are dual bases of $H^*(W/\!\!/\mathbf{G})$ with respect to $\langle\ ,\ \rangle$. Here $\mathbb{1}$ denotes the cohomology class dual to the fundamental cycle.
- $\mathrm{Eff}(W, \mathbf{G}, \theta)$ denotes the semigroup of numerical classes $\beta \in \mathrm{Pic}([W/\mathbf{G}])^\vee$ represented by θ-quasimaps with possibly disconnected domain. (Note that $\mathrm{Eff}(W, \mathbf{G}, \theta)$ is in general bigger than the cone of effective curves in $W/\!\!/\mathbf{G}$.)
- $\Lambda \cong \mathbb{Q}[[q]]$ denotes the Novikov ring of the theory, that is, the q-adic completion of the semigroup ring $\mathbb{C}[\mathrm{Eff}(W, \mathbf{G}, \theta)]$, $\beta \leftrightarrow q^\beta$.

4.1. *S*-operators.

For $\delta_i \in H^*(W/\!\!/\mathbf{G})$ and integers $a_i \geq 0$, put

$$\langle\langle \delta_1\psi_1^{a_1}, \ldots, \delta_k\psi_k^{a_k} \rangle\rangle_{g,k}^{\varepsilon} = \sum_{\beta \in \mathrm{Eff}(W,\mathbf{G},\theta)} \sum_{m \geq 0} \frac{q^\beta}{m!} \langle \delta_1\psi_1^{a_1}, \ldots, \delta_k\psi_k^{a_k}, t, \ldots, t \rangle_{g,k+m,\beta}^{\varepsilon}.$$

It is a formal function of $t = \sum_{i=1}^s t_i\gamma_i \in H^*(W/\!\!/\mathbf{G})$.

Define, for $\gamma \in H^*(W/\!\!/\mathbf{G}, \Lambda)$ and a formal variable z,

$$S_t^\varepsilon(z)(\gamma) := \sum_{i=1}^s \gamma_i \langle\langle \frac{\gamma^i}{z - \psi}, \gamma \rangle\rangle_{0,2}^{\varepsilon}(t).$$

Here $\psi = \psi_1$ and the right-hand side is interpreted as usual by expanding $1/(z-\psi)$ as a geometric series in ψ/z. By convention, $\langle \frac{\gamma^i}{z-\psi}, \gamma \rangle_{0,2,0}^{\varepsilon} = \langle \gamma^i, \gamma \rangle$. We think of S_t^ε as a family (parametrized by t) of operators on $H^*(W/\!\!/\mathbf{G}, \Lambda)$. When the variable z is understood we drop it from the notation.

In Gromov-Witten theory, the operator S_t^∞ is well-known. Its matrix is the (inverse of) a fundamental solution for the quantum differential equation. Furthermore, by the string equation for Gromov-Witten invariants, $S_t^\infty(\mathbb{1})$ coincides with Givental's (big) J-function of $W/\!\!/\mathbf{G}$ (we will come back to J-functions in the next subsection). The operator S_t^∞ determines the entire genus zero sector of the Gromov-Witten theory of $W/\!\!/\mathbf{G}$ by a standard reconstruction procedure, essentially due to Dubrovin [23]. As shown in [12], the same reconstruction works for ε-quasimap invariants for all $\varepsilon \geq 0+$. The key point where a new idea is needed is the proof of the following result, which reconstructs invariants with *two* descendant insertions.

Theorem 4.1. *Let z, w be formal variables and define*

$$V_t^\varepsilon(z,w) := \sum_{i,j=1}^{s} \gamma_i \otimes \gamma_j \langle\langle \frac{\gamma^i}{z-\psi}, \frac{\gamma^j}{w-\psi} \rangle\rangle_{0,2}^{\varepsilon}(t),$$

where $[\Delta] = \sum_{i=1}^{s} \gamma_i \otimes \gamma^i \in H^*(W/\!\!/\mathbf{G}) \otimes H^*(W/\!\!/\mathbf{G})$ is the cohomology class of the diagonal and the convention

$$\sum_{i,j=1}^{s} \gamma_i \otimes \gamma_j \langle \frac{\gamma^i}{z-\psi}, \frac{\gamma^j}{w-\psi} \rangle_{0,2,0}^{\varepsilon} = \frac{[\Delta]}{z+w}$$

is made for the unstable term in the double bracket. Then

$$V_t^\varepsilon = \frac{S_t^\varepsilon(z) \otimes S_t^\varepsilon(w)([\Delta])}{z+w}.$$

The usual - and very easy - argument that proves the above theorem in Gromov-Witten theory (see [27, item (4) on p.117]) requires the string equation and therefore does not extend to stability parameters $0+ \leq \varepsilon \leq 1$. The new proof from [12] is uniform for all values of ε.

4.2. Wall-crossing for S-operators.

The most general wall-crossing formula in genus zero applies to the operators S_t^ε, see [11, Theorem 7.3.1]. We state here a slightly more special case.

Theorem 4.2. *Assume that there is an action by a torus $\mathbf{T}$ on W, commuting with the action of $\mathbf{G}$, and such that the induced $\mathbf{T}$-action on $W/\!\!/\mathbf{G}$ has isolated fixed points. For every $\varepsilon \geq 0+$*

$$S_t^\varepsilon(\mathbb{1}) = S_{\tau^\varepsilon(t)}^\infty(\mathbb{1}),$$

where the (invertible) transformation $\tau^\varepsilon(t)$ is the series of primary ε-quasimap invariants

$$\begin{aligned} \tau^\varepsilon(t) &= \sum_{i=1}^{s} \gamma_i \langle\langle \gamma^i, \mathbb{1} \rangle\rangle_{0,2}^{\varepsilon}(t) - \mathbb{1} \\ &= t + \sum_{i=1}^{s} \gamma_i \sum_{\beta \neq 0} \sum_{m \geq 0} \frac{q^\beta}{m!} \langle \gamma^i, \mathbb{1}, t, \dots, t \rangle_{0,2+m,\beta}^{\varepsilon}. \end{aligned}$$

Moreover, the same statement holds for E-twisted theories, where E is any convex $\mathbf{G}$-representation.

A $\mathbf{G}$-representation is called convex if the $\mathbf{G}$-equivariant bundle $W \times E$ on W is generated by $\mathbf{G}$-equivariant sections. By twisted theories in the last statement we mean that the twisting is by the top Chern class, in the sense of Coates - Givental [18], as extended for quasimap invariants in [14, §6.2]. The twisting vector bundle $\mathcal{E}$ on $W/\!\!/\mathbf{G}$ is descended from the representation E.

Note that no positivity assumptions are made in Theorem 4.2 on $(W, \mathbf{G}, \theta)$, or on $(W, E, \mathbf{G}, \theta)$ in the twisted case, and also that no assumption is made on the 1-dimensional orbits of the $\mathbf{T}$ action on $W/\!\!/\mathbf{G}$.

Theorem 4.2 applies to essentially every example listed earlier: toric manifolds, flag manifolds, some (but not all) Nakajima quiver varieties, and local targets over them all admit torus actions with the required property. Of course, the statement is conjectured to hold without the existence of a torus action with isolated fixed points, see [11, Conjecture 6.1.1]. In fact, the part of the Theorem involving twisted theories already covers such targets. This is because the E-twisted quasimap invariants give (almost all of) the genus zero quasimap invariants of the zero-locus of a regular section of the bundle $\mathcal{E} = W^{ss} \times_{\mathbf{G}} E$ and this zero-locus generally is not $\mathbf{T}$-invariant.

4.3. J^ε-functions and Birkhoff factorization. Recall first the big J-function of Gromov-Witten theory:

$$\begin{aligned} J^\infty(q,t,z) &= \mathbb{1} + \frac{t}{z} + \sum_i \gamma_i \langle\!\langle \frac{\gamma^i}{z(z-\psi)} \rangle\!\rangle^\infty_{0,1}(t) \\ &= \mathbb{1} + \frac{t}{z} + \sum_{\beta,k} \frac{q^\beta}{k!} (ev_1)_* \frac{[\overline{M}_{0,1+k}(W/\!\!/\mathbf{G},\beta)]^{\mathrm{vir}} \cap \prod_{j=2}^{1+k} ev_j^* t}{z(z-\psi)} \end{aligned}$$

(the last sum is over $(\beta,k) \neq (0,0),(0,1)$). We want to extend it to all $\varepsilon \geq 0+$. The problem is that the spaces $\mathrm{Q}^\varepsilon_{0,1}(W/\!\!/\mathbf{G},\beta)$ do not exist for $\varepsilon \leq \frac{1}{\beta(L_\theta)}$. To resolve it we use the interpretation of the J-function as a sum of certain virtual localization residues for the natural $\mathbb{C}^*$-action on the Gromov-Witten *graph spaces* $\overline{M}_{0,k}(W/\!\!/\mathbf{G} \times \mathbb{P}^1, (\beta,1))$.

Specifically, for *all* $0+ \leq \varepsilon$, $k \geq 0$, we have the *quasimap graph space*

$$QG^\varepsilon_{0,k,\beta}(W/\!\!/\mathbf{G}) = \{\big((C,\{p_i\}),P,u,\varphi\big) \mid \varphi : C \to \mathbb{P}^1, \varphi_*[C] = [\mathbb{P}^1]\}.$$

This is the moduli space of (genus zero, k-pointed) ε-stable quasimaps whose domain curve contains a component C_0 which is a parametrized $\mathbb{P}^1$. The ampleness part of the ε-stability condition involves only $\overline{C \setminus C_0}$, while the length condition remains the same. These spaces are defined for all $k \geq 0$ and the analogue of Theorem 3.2 holds for them. For toric targets and $\varepsilon = 0+$ they were introduced in [10], the general case is in [14].

The $\mathbb{C}^*$-action on $\mathbb{P}^1$ induces an action on $QG^\varepsilon_{0,k,\beta}(W/\!\!/\mathbf{G})$. Let z denote the equivariant parameter.

Consider the fixed locus F_0 of quasimaps for which all markings and the entire degree β are over $0 \in \mathbb{P}^1 \cong C_0$. There are two cases:

- $k \geq 1$, or $\varepsilon > \frac{1}{\beta(L_\theta)}$. Then $F_0 \cong \mathrm{Q}^\varepsilon_{0,1+k}(W/\!\!/\mathbf{G},\beta)$ with its canonical virtual class and $\mathrm{e}_{\mathbb{C}^*}(N^{\mathrm{vir}}) := \mathrm{e}_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_0/QG^\varepsilon_{0,k,\beta}(W/\!\!/\mathbf{G})}) = z(z-\psi)$. We also have the evaluation map $ev = ev_1 : F_0 \to W/\!\!/\mathbf{G}$.

- $k=0$ and $\varepsilon \leq \frac{1}{\beta(L_\theta)}$. Then $F_0 = \{(\mathbb{P}^1, P, u)\}$, with u having a base point of (maximal) length $\beta(L_\theta)$ at $0 \in \mathbb{P}^1$. We define $ev : F_0 \to W/\!\!/\mathbf{G}$ by taking evaluation at the generic point of $\mathbb{P}^1$. In this case $\mathrm{e}_{\mathbb{C}^*}(N^{\mathrm{vir}})$ changes with β.

Now for each $\varepsilon \geq 0+$ we define the big J^ε-function by

$$\begin{aligned} J^\varepsilon(q,t,z) &:= \sum_{\beta,k\geq 0} \frac{q^\beta}{k!} ev_* \mathrm{Res}_{F_0}\big([QG^\varepsilon_{0,k,\beta}(W/\!\!/\mathbf{G})]^{\mathrm{vir}} \cap \prod_{j=1}^k ev_j^* t\big) \\ &= \mathbb{1} + \frac{t}{z} + \sum_{0<\beta(L_\theta)\leq 1/\varepsilon} q^\beta ev_* \frac{[F_0]}{\mathrm{e}_{\mathbb{C}^*}(N^{\mathrm{vir}})} \\ &+ \sum_{\beta,k} \frac{q^\beta}{k!} (ev_1)_* \frac{[\mathrm{Q}^\varepsilon_{0,1+k}(W/\!\!/\mathbf{G},\beta)]^{\mathrm{vir}} \cap \prod_{j=2}^{1+k} ev_j^* t}{z(z-\psi)}. \end{aligned}$$

The *small* J^ε-function is by definition the specialization at $t=0$,

$$J^\varepsilon_{sm}(q,z) := J^\varepsilon(q,0,z).$$

For the asymptotic stability $\varepsilon = 0+$ we have the *small I-function*

$$I_{sm}(q,z) = J^{0+}(q,0,z) = \mathbb{1} + \sum_{\beta\neq 0} q^\beta ev_* \frac{[F_0]}{\mathrm{e}_{\mathbb{C}^*}(N^{\mathrm{vir}})}.$$

When $W/\!\!/\mathbf{G}$ is a nonsingular toric variety, or a complete intersections in a toric variety, the small I-function is (essentially up to an exponential factor) the cohomology valued hypergeometric q-series introduced by Givental, see [26].

Closed expressions for I_{sm} are known also for many non-abelian GIT quotients: flag manifolds of classical type, zero loci of sections of homogeneous bundles in them, local targets over them, the Hilbert scheme of points in $\mathbb{C}^2$ ([4, 5, 15, 16]).

In general, the big J^ε-function and the operator S^ε_t are related by "Birkhoff factorization". This is the content of the following Theorem.

Theorem 4.3. ([11]) *For any GIT target and any* $\varepsilon \geq 0+$

$$J^\varepsilon(q,t,z) = S^\varepsilon_t(P^\varepsilon(q,t,z))$$

where $P^\varepsilon(q,t,z)$ *is a* power series *in* z. *(In fact,* P^ε *is naturally a generating function of* $\mathbb{C}^*$*-equivariant graph space integrals, see [11, §5.4].)*

4.4. The case of semi-positive targets.

The triple $(W,\mathbf{G},\theta)$ is called *semi-positive* if

$$\beta(\det(T_W)) \geq 0$$

for every $\beta \in \mathrm{Eff}(W,\mathbf{G},\theta)$. Here T_W is the (virtual) tangent bundle of the lci $\mathbf{G}$-variety W, viewed as an element in the equivariant K-group $K^0_{\mathbf{G}}(W)$. We note that semi-positivity implies that the anti-canonical class of a projective $W/\!\!/\mathbf{G}$ is nef, but the converse need not be true.

The Birkhoff Factorization in Theorem 4.3 simplifies drastically for semi-positive targets. If $(W, \mathbf{G}, \theta)$ is semi-positive, easy dimension counting arguments show that for every $\varepsilon \geq 0+$ the function J^ε contains no positive powers of z. Hence we have the asymptotic expansions

$$J^\varepsilon_{sm}(q,z) = J^\varepsilon_0(q)\mathbb{1} + \frac{J^\varepsilon_1(q)}{z} + O(1/z^2),$$

$$J^\varepsilon(q,t,z) = J^\varepsilon_0(q)\mathbb{1} + \frac{t + J^\varepsilon_1(q)}{z} + O(1/z^2).$$

In particular, we have

$$I_{sm}(q,z) = I_0(q)\mathbb{1} + I_1(q)\frac{1}{z} + O(1/z^2),$$

defining the q-series $I_0(q)$ and $I_1(q)$. They satisfy $I_0(q) = 1 + O(q) \in \Lambda$ and $I_1 \in qH^{\leq 2}(W/\!\!/\mathbf{G}, \Lambda)$. For $\varepsilon > 0$, the coefficients $J^\varepsilon_0(q)$ and $J^\varepsilon_1(q)$ are polynomial truncations of the series I_0 and I_1. Note that since there are explicit closed formulas for I_{sm} in almost all examples, the series $I_0(q)$ and $I_1(q)$ are also explicit.

It follows that Theorem 4.3 specializes to the following Corollary (a very special case of this result is due to [19], by different methods).

Corollary 4.4. ([11]) *Let $(W, \mathbf{G}, \theta)$ be semi-positive and let $\varepsilon \geq 0+$ be arbitrary. Then the J-function and the S-operator are related by*

$$S^\varepsilon_t(\mathbb{1}) = \frac{J^\varepsilon(q,t,z)}{J^\varepsilon_0(q)}.$$

The transformation $\tau_\varepsilon(t) = \sum_{i=1}^s \gamma_i \langle\!\langle \gamma^i, \mathbb{1} \rangle\!\rangle^\varepsilon_{0,2}(t) - \mathbb{1}$ satisfies

$$\tau_\varepsilon(t) = \frac{t + J^\varepsilon_1(q)}{J^\varepsilon_0(q)},$$

and in particular

$$\sum_{i=1}^s \gamma_i \sum_{\beta \neq 0} q^\beta \langle \gamma^i, \mathbb{1} \rangle^\varepsilon_{0,2,\beta} = \frac{J^\varepsilon_1(q)}{J^\varepsilon_0(q)}.$$

Combining Theorem 4.2 with Corollary 4.4 gives the following Corollary.

Corollary 4.5. ([11]) *Assume $(W, \mathbf{G}, \theta)$ is semi-positive and there is a $\mathbf{T}$-action on $W/\!\!/\mathbf{G}$ with isolated fixed points. Then*

$$J^\infty\left(q, \frac{t + J^\varepsilon_1(q)}{J^\varepsilon_0(q)}, z\right) = \frac{J^\varepsilon(q,t,z)}{J^\varepsilon_0(q)}.$$

The same is true for E-twisted theories on $W/\!\!/\mathbf{G}$, where E is a convex $\mathbf{G}$-representation such that, for all θ-effective β,

$$\beta(\det(T_W)) - \beta(W \times \det(E)) \geq 0.$$

Let $\varepsilon = 0+$. After making $t = 0$ in the last Corollary and applying the string and divisor equations in the GW side, we obtain the usual formulation of the genus zero Mirror Theorem for the small J-function of Gromov-Witten theory

$$e^{\frac{1}{z}\frac{I_1(q)}{I_0(q)}} J^{\infty}_{sm}(Q,z) = I_{sm}(q,z)$$

after the change of variable $Q^\beta = q^\beta e^{\int_\beta \frac{I_1(q)}{I_0(q)}}$. For Calabi-Yau complete intersections in toric varieties, this change of variables is precisely the mirror map obtained from the solutions to the Picard-Fuchs equations associated to the mirror manifolds. Note that by the last equation in Corollary 4.4 the mirror map acquires a geometric interpretation in terms of two-point primary $(0+)$-quasimap invariants with a fundamental class insertion, as suggested by Jinzenji [33].

Hence the genus zero wall-crossing formula in Theorem 4.2 generalizes the mirror theorems as follows:

- from abelian to non-abelian quotients
- from the small to the big phase space
- from $\varepsilon = 0+$ to all ε
- from semi-positive GIT triples to all such triples.

4.5. Wall-crossing for J^ε-functions in the general case.

Without the semi-positivity assumption the relation between $J^\varepsilon(q,t,z)$ and $J^\infty(q,t,z)$ is more complicated than the one given by Corollary 4.5. The most concise formulation is given by the following Conjecture [11, Conjecture 6.4.2].

Conjecture 4.6. *For all GIT triples $(W, \mathbf{G}, \theta)$ and all stability parameters $\varepsilon \geq 0+$ the function $J^\varepsilon(q,t,z)$ is on the Lagrangian cone of the Gromov-Witten theory of $W/\!\!/\mathbf{G}$.*

Recall that for a general target X Givental introduced a formalism which encodes the genus zero sector of the Gromov-Witten theory of X via an overruled Lagrangian cone in an appropriate infinite-dimensional symplectic vector space, see [28, 18]. The Lagrangian cone is generated by the big J-function (this statement is another formulation of the Dubrovin reconstruction mentioned earlier). The conjecture then implies that $J^\infty(q, \tau_{\infty,\varepsilon}(q,t), z)$ is a linear combination of the derivatives $\partial_{t_i} J^\varepsilon(q,t,z)$ with uniquely determined coefficients (depending on q, t, and z) and unique change of variables $t \mapsto \tau_{\infty,\varepsilon}(q,t)$.

Theorem 4.7. ([11]) *Assume there is a $\mathbf{T}$-action on W such that the induced action on $W/\!\!/\mathbf{G}$ has isolated fixed points and isolated 1-dimensional orbits. Then Conjecture 4.6 holds true.*

4.6. Big $\mathbb{I}$-functions.

The results described so far in this section elucidate the relationship between quasimap and GW invariants of $W/\!\!/\mathbf{G}$ in genus zero. However, if one is primarily interested in calculating GW invariants, the applicability of these results is restricted only to invariants with (descendant) insertions at one marking. This is because only for the small I-function one can write down explicit closed formulas. In general, quasimap invariants with two or more insertions are equally difficult to determine for all values of the stability parameter ε. To improve the situation the authors have introduced in [13] a new version of big $\mathbb{I}$-function of $(W, \mathbf{G}, \theta)$ by considering a theory of (0+)-stable quasimaps with infinitesimally weighted markings. We conjectured that this function lies on the Lagrangian cone of the Gromov-Witten theory of $W/\!\!/\mathbf{G}$. Arguments parallel to the ones in the unweighted case are used to prove the following Theorem.

Theorem 4.8. ([13]) *Let $(W, \mathbf{G}, \theta)$ be a GIT triple. Assume there is a $\mathbf{T}$-action on W such that the induced action on $W/\!\!/\mathbf{G}$ has isolated fixed points and isolated 1-dimensional orbits. Then the big $\mathbb{I}$-function associated to $(W, \mathbf{G}, \theta)$ is on the Lagrangian cone of the Gromov-Witten theory of $W/\!\!/\mathbf{G}$. Furthermore, if E is a convex $\mathbf{G}$-representation, then the E-twisted $\mathbb{I}$ is on the E-twisted Lagrangian cone of $W/\!\!/\mathbf{G}$.*

As a consequence, the big J function of the (E-twisted) Gromov-Witten theory of $W/\!\!/\mathbf{G}$ is obtained from $\mathbb{I}$ via the Birkhoff factorization procedure of Coates and Givental [18]. The advantage is that one can calculate again explicit closed formulas for this new big $\mathbb{I}$-function in many cases. In [13] it is explained how to do so for toric varieties and for complete intersections in them. For example, if $\mathbb{C}^{n+1}/\!\!/_{\mathrm{id}}\mathbb{C}^* = \mathbb{P}^n$ is the standard GIT presentation of the projective space and $E = \mathbb{C}_{l(\mathrm{id})}$ is the 1-dimensional $\mathbb{C}^*$-representation with weight $l \in \mathbb{Z}_{>0}$, then one finds

$$\mathbb{I}^E_{\mathbb{C}^{n+1}/\!\!/\mathbb{C}^*}(t) = \sum_{d=0}^{\infty} q^d \frac{\exp(\sum_{i=0}^n t_i (H+dz)^i/z)}{\prod_{k=1}^d (H+kz)^{n+1}} \prod_{k=0}^{ld} (lH+kz),$$

where H is the hyperplane class and $t = \sum_{i=0}^n t_i H^i$ is the general element of $H^*(\mathbb{P}^n, \mathbb{Q})$.

Observe that if we denote by $t_{sm} = t_0 \mathbb{1} + t_1 H$ the restriction of t to the small parameter space $H^0(\mathbb{P}^n, \mathbb{Q}) \oplus H^2(\mathbb{P}^n, \mathbb{Q})$, then

$$\mathbb{I}^E_{\mathbb{C}^{n+1}/\!\!/\mathbb{C}^*}(t_{sm}) = \exp(\frac{t_0 \mathbb{1} + t_1 H}{z}) \sum_{d=0}^{\infty} q^d \exp(dt_1) \frac{\prod_{k=0}^{ld}(lH+kz)}{\prod_{k=1}^d (H+kz)^{n+1}},$$

which is precisely Givental's small I-function of a hypersurface of degree l in $\mathbb{P}^n$ (and differs from the function I_{sm} from §4.3 by the overall exponential factor $\exp(\frac{t_0\mathbb{1}+t_1 H}{z})$ and the change $q \mapsto q\exp(t_1)$).

Remark 4.9. The results described in §4.2 - §4.6 above have been extended in [6] to the case of "orbifold GIT targets", that is, to the case when $[W^{ss}/\mathbf{G}]$ is a nonsingular Deligne-Mumford stack. A result related to Theorem 4.8 has been obtained earlier by Woodward, [53, Theorem 1.6].

5. Higher genus wall-crossing for semi-positive targets

In this section we discuss the wall-crossing formulas for higher genus ε-quasimap descendant invariants in the case of semi-positive triples $(W, \mathbf{G}, \theta)$.

Let

$$\mathbf{t}(\psi) := t_0 + t_1\psi + t_2\psi^2 + t_3\psi^3 + \dots,$$

with $t_j = \sum_i t_{ji}\gamma_i \in H^*(W/\!\!/\mathbf{G}, \mathbb{Q})$ general cohomology classes.

By definition, the genus g, ε-descendant potential of $(W, \mathbf{G}, \theta)$ is

$$F_g^\varepsilon(\mathbf{t}) := \sum_{\beta \in \mathrm{Eff}(W,\mathbf{G},\theta)} \sum_{m \geq 0} \frac{q^\beta}{m!} \langle \mathbf{t}(\psi_1), \mathbf{t}(\psi_2), \dots \mathbf{t}(\psi_m) \rangle^\varepsilon_{g,m,\beta}.$$

As usual, we omit from the sum the unstable terms corresponding to $(g, m, \beta, \varepsilon)$ for which the moduli spaces are not defined.

Conjecture 5.1. ([12]) *For a semi-positive triple $(W, \mathbf{G}, \theta)$, and every $\varepsilon \geq 0+$*

$$(J_0^\varepsilon(q))^{2g-2} F_g^\varepsilon(\mathbf{t}(\psi)) = F_g^\infty\left(\frac{\mathbf{t}(\psi) + J_1^\varepsilon(q)}{J_0^\varepsilon(q)}\right). \tag{5.0.1}$$

Further, for every $\varepsilon_1 \neq \varepsilon_2$

$$(J_0^{\varepsilon_1}(q))^{2g-2} F_g^{\varepsilon_1}(J_0^{\varepsilon_1}(q)\mathbf{t}(\psi) - J_1^{\varepsilon_1}(q)) = (J_0^{\varepsilon_2}(q))^{2g-2} F_g^{\varepsilon_2}\left(J_0^{\varepsilon_2}(q)\mathbf{t}(\psi) - J_1^{\varepsilon_2}(q)\right). \tag{5.0.2}$$

To be precise, in the case $g = 0$ the equalities are conjectured to hold modulo terms of degree ≤ 1 in the coordinates t_{ji} (but see [12, Remark 3.1.3] for an explanation on how to extend the statement to an equality up to constants).

Note that the (a priori stronger) wall-crossing formula (5.0.2) follows from (5.0.1).

Considering the Taylor coefficients on both sides gives the following equivalent formulation of (5.0.1): If $2g-2+k \geq 0$, then for arbitrary $\delta_1, \dots \delta_k \in H^*(W/\!\!/\mathbf{G}, \mathbb{Q})$ and integers $a_1, \dots, a_k \geq 0$,

$$(J_0^\varepsilon(q))^{2g-2+k} \sum_\beta q^\beta \langle \delta_1\psi_1^{a_1}, \dots, \delta_n\psi_k^{a_k} \rangle^\varepsilon_{g,k,\beta} =$$
$$\sum_\beta q^\beta \sum_{m=0}^\infty \frac{1}{m!} \left\langle \delta_1\psi_1^{a_1}, \dots, \delta_k\psi_k^{a_k}, \frac{J_1^\varepsilon(q)}{J_0^\varepsilon(q)}, \dots, \frac{J_1^\varepsilon(q)}{J_0^\varepsilon(q)} \right\rangle^\infty_{g,k+m,\beta}.$$

Combining Corollary 4.5 with reconstruction for ε-quasimap invariants proves the Conjecture in genus zero.

Theorem 5.2. ([12]) *Let $(W, \mathbf{G}, \theta)$ be semi-positive. Assume there is an action by a torus $\mathbf{T}$, such that the fixed points of the induced $\mathbf{T}$-action on $W/\!\!/\mathbf{G}$ are* *isolated. Then the $g = 0$ case of Conjecture 5.1 holds. Moreover, if E is a convex* $\mathbf{G}$*-representation such that $\beta(\det(T_W)) - \beta(W \times \det(E)) \geq 0$ for all θ-effective β, then the conjecture also holds at $g = 0$ for the E-twisted ε-quasimap theories of* $W /\!/ \mathbf{G}$.

A more convincing piece of evidence for the validity of the Conjecture is provided by the following result:

Theorem 5.3. ([12]) *Let X be a nonsingular quasi-projective toric variety of dimension n, obtained as the GIT quotient of a semi-positive triple* $(\mathbb{C}^{n+r}, (\mathbb{C}^*)^r, \theta)$. *Then Conjecture 5.1 holds for X.*

It is easy to see that toric varieties (in any semi-positive GIT presentation) have $I_0(q) = 1$ (and hence $J_0^\varepsilon = 1$ for all ε). When X is a nonsingular and projective toric Fano and we take its "standard" GIT presentation (as considered e.g. in [10]), then $I_1(q) = 0$ as well. Hence we obtain the following

Corollary 5.4. *If X is a nonsingular projective Fano toric variety, then its quasimap invariants (for the standard GIT presentation) are independent on ε:*

$$F_g^\varepsilon(\mathbf{t}(\psi)) = F_g^\infty(\mathbf{t}(\psi)), \ \ \forall \varepsilon \geq 0+\, .$$

The first statement of the kind in the Corollary was established by Marian - Oprea - Pandharipande [41] for $W /\!/ \mathbf{G}$ a Grassmannian and for $\varepsilon = 0+$. Their result was extended to all ε in [51] by Toda.

Remark 5.5.

1. The most interesting case covered by Theorem 5.3 is that of toric Calabi-Yau targets. For 3-folds, our theorem says that $F_g^{0+}|_{\mathbf{t}(\psi)=0}$ is *equal* to the B-model genus g pre-potential, expanded near a large complex structure point for the mirror of X.

2. The arguments proving Theorem 5.3 also apply to show that the higher genus wall-crossing of Conjecture 5.1 holds for some non-abelian local Calabi-Yau targets, namely local Grassmannians, and in fact local type A flag manifolds, see [12, Theorem 1.3.4].

3. The remaining challenge is to prove Conjecture 5.1 for compact Calabi-Yau targets at $g \geq 1$.

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