Homework #3 for MATH 5345H: Introduction to Topology

September 24, 2018

Due Date: Friday 28 September in class.

1. Let X be a set, and let T be a topology on X in which the singleton $\{x\} \in T$ for each $x \in X$. Show that T is the discrete topology on X.

2. Let $\mathbb{R}[x_1, \ldots, x_n]$ denote the set of polynomials in n variables $x_1, \ldots, x_n$ whose coefficients lie in $\mathbb{R}$. So, for instance, $x_1 - 3x_2^2 + \sqrt{2}x_7^4 \in \mathbb{R}[x_1, \ldots, x_9]$, but neither $\frac{x_1}{x_2}$ nor ix_5^3 is an element of this set of polynomials.

   For a subset $S \subseteq \mathbb{R}[x_1, \ldots, x_n]$, write $V(S) \subseteq \mathbb{R}^n$ to be the set

   $$\begin{aligned} V(S) &= \{(x_1, \ldots, x_n) \mid f(x_1, \ldots, x_n) = 0, \forall f \in S\} \\ &= \bigcap_{f \in S} \{(x_1, \ldots, x_n) \mid f(x_1, \ldots, x_n) = 0\}. \end{aligned}$$

   Let $U(S) = \mathbb{R}^n \backslash V(S)$. We will show that the collection $T_Z = \{U(S),\ S \subseteq \mathbb{R}[x_1, \ldots, x_n]\}$ forms a topology on $\mathbb{R}^n$, called the *Zariski topology.*

   (a) For any real number r (such as $r = 0$ or $r = 1$), write r for the constant polynomial r. Show that $V(\{0\}) = \mathbb{R}^n$.

   (b) Show that $V(\{1\}) = \emptyset$.

   (c) Show that, for any indexing set J,

   $$V(\bigcup_{j \in J} S_j) = \bigcap_{j \in J} V(S_j)$$

   (d) For any two sets $S, T \subseteq \mathbb{R}[x_1, \ldots, x_n]$, define

   $$ST := \{f \cdot g \mid f \in S, g \in T\}.$$

   Show that $V(ST) = V(S) \cup V(T)$.

   (e) Show that T_Z is a topology on $\mathbb{R}^n$.

(f) Fix $n = 1$, and show that for any set $S \subseteq \mathbb{R}[x_1]$, $V(S)$ is finite. Conversely, let $F \subset \mathbb{R}$ be any finite set. Find a set $T \subseteq \mathbb{R}[x_1]$ with $V(T) = F$.

(g) Show that the Zariski topology on $\mathbb{R}^1$ is equal to the finite complement topology.

3. Show that if $\mathcal{B}$ is a basis for a topology on X, then the topology generated by $\mathcal{B}$ equals the intersection of all topologies on X that contain $\mathcal{B}$. Is the same true for a subbasis?