Homework #8 for MATH 5345H: Introduction to Topology

November 19, 2018

Due Date: Monday 26 November in class.

Focus on writing: Writing strong mathematical proofs is just as much about quality writing as it is about quality content. Over the next few weeks, in addition to writing up solutions to your problem sets as usual, I will ask you to focus intently on improving one aspect of your proof writing skills.

Structuring quality proofs takes careful thought; proofs should not be written in stream-of-conscience form. Writing out one's train of thoughts is a great way to build a proof sketch, and it needs to be honed from there into an argument that follows a logical roadmap. Writing proofs with an intended structure in mind also improves the concision of your proofs.

Before writing your solution to Problem 4, I want you two write an outline to your argument that details the purpose of each section.

For example, a problem might ask you to prove that the group of integers is isomorphic to the group of even integers. An appropriate outline might be the following:

1. Define a map $\varphi : \mathbb{Z} \to 2\mathbb{Z}$
2. Prove φ is a bijection
    (a) Prove φ is injective
    (b) Prove φ is surjective
3. Prove φ is a homomorphism

After constructing your outline, you should should write your proof to follow it.

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1. Give $\mathbb{Q}$ the subspace topology of $\mathbb{R}$. Is it locally compact? Prove your answer.
2. Let X be a connected, locally compact, Hausdorff space. Show that X^* (the one-point compactification of X) is connected if and only if X is *not* compact.
3. Recall the wedge sum from previous homeworks: if A and B are two topological spaces, $a_0 \in A$ and $b_0 \in B$, then $A \vee B$ is the quotient space of $A \coprod B$ under the relation $a_0 \sim b_0$ (the points a_0 and b_0 are usually called *basepoints* of A and B).

(a) Let X and Y be locally compact, Hausdorff spaces, and let X^*, Y^* be their one point compactifications. Show that there is a homeomorphism

$$(X \coprod Y)^* \cong X^* \vee Y^*$$

where the basepoints of X^* and Y^* are taken to be the points at infinity.

(b) Describe the one-point compactification of the space X defined as

$$X := \{(x, y, z) \in \mathbb{R}^3 \mid y = 0 = z, \text{ OR } x = 1 = z\}.$$

4. Let $S = [0, 1] \times [0, 1]$ be the unit square, and define an equivalence relation $\sim$ on S generated[1] by the formulas

$$(x, 0) \sim (x, 1) \text{ and } (0, y) \sim (1, y).$$

Let T be the *torus* in $\mathbb{R}^4 = \mathbb{C}^2$:

$$T := \{(z, w) \in \mathbb{C}^2 \ s.t. \ ||z|| = 1 = ||w||\}$$

Construct a homeomorphism $f : (S/\sim) \to T$ and prove that it is a homeomorphism.

[1]Here, "generated" means that since these formulas are not symmetric, reflexive, or transitive, $\sim$ is defined to be the weakest equivalence relation which implies these formulas. E.g., one adds $(x, 1) \sim (x, 0)$, and so on.