Homework #9 for MATH 5345H: Introduction to Topology

November 28, 2018

Due Date: Monday 3 December in class.

Focus on writing: Writing strong mathematical proofs is just as much about quality writing as it is about quality content. Over the next few weeks, in addition to writing up solutions to your problem sets as usual, I will ask you to focus intently on improving one aspect of your proof writing skills.

Oftentimes, the logic behind a mathematical argument is neither straightforward nor immediately apparent. To clarify the argument for your audience, it may be insightful to provide an example or counterexample to the statement of the problem that illustrates the underlying idea. This week, I will provide for you a mathematical statement in problem 1(c). If the statement is correct, you should create an illustrative example that provides some intuition for the reader about why the statement is true, and follow up with an actual proof of the statement. If the statement is false, you should exhibit a counterexample that demonstrates the falsehood.

Here is an example of such an assignment. As setup: we say that a pair of points $x, y \in X$ in a topological space X are *topologically indistinguishable* if they have exactly the same open neighborhoods. If this is not the case, they are *topologically distinguishable*.

Statement: If X is a topological space in which every pair of points is topologically distinguishable, then X is Hausdorff.

Illustrative Example: The statement is false. Recall that for any pair of points $\{x, y\}$ beloinging to a Hausdorff space X, there exists an open neighborhood about each point that does not contain the other. Consider a finite topological space $X = \{x, y\}$ equipped with open sets $\emptyset$, $\{x\}$, and X. Because there exists an open set containing x that does not contain y, the points x and y are distinguishable. The only open set containing y is X, which also contains x, so X cannot be Hausdorff.

1. Let X be a metric space. Recall that X is *complete* if every Cauchy sequence in X converges (recall further that $\{x_n\}$ is *Cauchy* if, for every $\epsilon > 0$, there exists an N such that if $m, n > N$, $d(x_m, x_n) < \epsilon$).

   Let X be a complete metric space and let $f : X \to X$ be a function satisfying the following condition: there is some $r \in [0, 1)$ such that for any points $x, y \in X$,

   $$d(f(x), f(y)) \leq rd(x, y).$$

   Such a map f is called a *contraction mapping*.

   (a) Show that, for any $x \in X$, the sequence

   $$x, f(x), f(f(x)), \cdots$$

   is Cauchy.

   (b) (The contraction mapping theorem) Using the previous problem, show that f has a unique fixed point (that is, there is a unique $x_0 \in X$ such that $f(x_0) = x_0$).

   (c) Consider the situation in which f is an *isometry*:

   $$d(f(x), f(y)) = d(x, y),$$

   (i.e., the case $r = 1$ of the previous setup). If X is assumed to be compact, must f have a fixed point? Either prove this result and give an illustrative example, or give an illustrative counterexample.

   (d) Suppose that X is a compact metric space and $f : X \to X$ is an isometry. Show that f is surjective.

   Hint: if $x \notin f(X)$, find an ϵ-ball around x missing $f(X)$, and consider once again the sequence

   $$x, f(x), f(f(x)), \cdots .$$

   Sequential compactness may help.

   (e) Using the previous problem, show that an isometry is a homeomorphism.

2. Let X be a metric space, equipped with the metric topology.

   (a) Show that if X is separable, then it is second countable.

   (b) Show that if X is Lindelöf, then it is second countable.