Midterm for MATH 5345H: Introduction to Topology

October 18, 2019

Due Date: Friday 25 October in class. You may use your book, notes, and old homeworks for this exam. When using results from any of these sources, please cite the result being used. Please explain all of your arguments carefully. Please do not communicate with other students about the exam. You are free to contact me with questions about the exam at any time.

1. Let $\mathbb{Q}$ denote the rational numbers.

   (a) Let T_{order} denote the order topology on $\mathbb{Q}$ with respect to the usual $<$, and let $T_{subspace}$ denote the subspace topology on $\mathbb{Q}$, as a subspace of $\mathbb{R}$ with the standard topology. Show that $T_{order} = T_{subspace}$.

   (b) Are the two topologies in the previous problem the same as the discrete topology? Why or why not?

2. Let (X, T) be a topological space. (X, T) is called an *Alexandroff space* if it has the property that arbitrary intersections of open sets are open (i.e., T is closed under arbitrary intersections, not just finite intersections). This problem will be about developing some of the basics of the topology of Alexandroff spaces.

   (a) Show that if X is finite, (X, T) is an Alexandroff space, regardless of the choice of T.

   (b) Show that $\mathbb{R}$ is *not* an Alexandroff space when equipped with the standard topology.

   (c) If (X, T) is an Alexandroff space, and $x \in X$, then by assumption the set

   $$U_x := \bigcap_{V \ni x} V \quad \text{where } V \text{ ranges over all open sets containing } x,$$

   is open; U_x is called the *minimal open set around x*. Show that the collection $\{U_x \mid x \in X\}$ forms a basis for the topology T.

   (d) Show that (X, T) is Hausdorff if and only if it is discrete.

   (e) Define a relation $\leq$ on X where $x \leq y$ if $U_x \subseteq U_y$. Prove that $\leq$ is a *preorder*; that is, $\leq$ is reflexive and transitive.

   (f) Show that $U_x = \{y \in X \mid y \leq x\}$.

   (g) In contrast, show that the closure of the singleton $\{x\} \subseteq X$ is $\overline{\{x\}} = \{y \in X \mid y \geq x\}$.

(h) Let (X, T) and (Y, S) be Alexandroff spaces, and $f : X \to Y$ a function. Show that f is continuous if and only if it is *order preserving*; that is, $x \leq x'$ implies[1] $f(x) \leq f(x')$.

3. Let $Y = \{a, b\}$ be a set with two elements, let X be any set, and let $P(X)$ be the power set of X (i.e., the set of subsets of X).

(a) Write $Y^X := \{f : X \to Y\}$ for the set of functions from X to Y. Show that the function

$$\alpha : Y^X \to P(X) \text{ given by } \alpha(f) = f^{-1}(\{a\})$$

is a bijection.

(b) Let $T_Y = \{\emptyset, \{a\}, Y\} \subseteq P(Y)$. I claim that T_Y is a topology on Y – this is easy to verify, and you do not need to do so.

Now assume, further, that X has a topology T_X. Prove that α restricts to a bijection

$$\alpha : \{f : X \to Y \mid f \text{ is continuous}\} \to T_X = \{U \subseteq X \mid U \text{ is open}\}.$$

That is: show that the bijection $\alpha : Y^X \to P(X)$ from the previous part defines a bijection from the subset of Y^X consisting of continuous maps to the subset of $P(X)$ consisting of open subsets of X (i.e., T_X).

[1] Here we're using $\leq$ to denote the preorder described above on both X and Y.