Homework #2 for MATH 8302: Manifolds and Topology II

February 19, 2018

Due Date: Monday 26 February in class.

1. We have constructed a ring structure on $H^*(X, R)$ for any topological space X and ring R using the cup product. It is not hard to show that if $f : X \to Y$ is a continuous map, then $f^* : H^*(Y, R) \to H^*(X, R)$ is a ring homomorphism.

   (a) Let $f : \mathbb{C}P^n \to \mathbb{C}P^n$ be a continuous map with the property that on H^2,
   $$f^* : H^2(\mathbb{C}P^n, \mathbb{Z}) \cong \mathbb{Z} \to H^2(\mathbb{C}P^n, \mathbb{Z}) \cong \mathbb{Z}$$
   is multiplication by d. Compute the self map f^* on $H^k(\mathbb{C}P^n, \mathbb{Z})$ for all k.

   (b) For f as in the previous problem, compute the Lefschetz number $\tau(f)$, and formulate and prove a criterion for when f has a fixed point.

   (c) A map f as in the previous problems is said to be *orientation reversing* if
   $$f^* : H^{2n}(\mathbb{C}P^n, \mathbb{Z}) \cong \mathbb{Z} \to H^{2n}(\mathbb{C}P^n, \mathbb{Z}) \cong \mathbb{Z}$$
   is multiplication by a negative number. Show that there are no orientation reversing self maps of $\mathbb{C}P^n$ if n is even.

2. (a) Suppose that $f : X \to Y$ is a smooth map, and let $F : X \to X \times Y$ be $F(x) = (x, f(x))$. Show that
   $$dF_x(v) = (v, df_x(v)).$$

   (b) Prove that the tangent space to the graph of f at the point $(x, f(x))$ is the graph of $df_x : T_xX \to T_{f(x)}Y$.

3. Let p be any homogenous polynomial in k variables (i.e., $p(tx_1, \ldots, tx_k) = t^d p(x_1, \ldots, x_k)$, where $d = \deg(p)$). Prove that the set
   $$V_a := \{x \in \mathbb{R}^k \mid p(x) = a\}$$
   is a $(k-1)$-dimensional submanifold of $\mathbb{R}^k$, provided that $a \neq 0$ (**Hint:** Euler has a relevant identity). Further, show that V_a and V_b are diffeomorphic, provided that $ab > 0$.