Homework #1 for MATH 8307: Algebraic Topology

February 17, 2015

Due Date: Wednesday 4 March in class.

1. Use the non-degeneracy of the cup-product pairing to inductively prove that there is an isomorphism of graded rings $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[c]/c^{n+1}$, where $\dim(c) = 2$. (Thanks, June!)

2. Let Σ_g be the closed, orientable surface of genus g; choose a fundamental class $[\Sigma_g] \in H_2(\Sigma_g)$. Recall that $H^1(\Sigma_g) \cong \mathbb{Z}^{2g}$.

   (a) Show that there is a basis for $H^1(\Sigma_g)$ in which the Poincaré duality pairing is represented by the matrix
   $$J_g := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$
   where I is the $g \times g$ identity matrix.

   (b) Define
   $$\Gamma_g := \{f : \Sigma_g \to \Sigma_g \mid f \text{ is a homotopy equivalence, and } f_*[\Sigma_g] = [\Sigma_g]\}/\sim$$
   where $f \sim h$ if they are homotopic. Show that Γ_g forms a group under composition (this is called the *mapping class group* of Σ_g, although it is not usually defined in quite this way).

   (c) Let $\mathrm{Sp}(2g, \mathbb{Z})$ denote the group of invertible, integral $2g \times 2g$ matrices M that preserve the bilinear form J_g (that is: $M^t J_g M = J_g$). This is called the (integral) *symplectic group*. Construct a homomorphism
   $$\varphi : \Gamma_g \to \mathrm{Sp}(2g, \mathbb{Z})$$
   using the induced map $f^* : H^1(\Sigma_g) \to H^1(\Sigma_g)$ of a homotopy equivalence.

   (d) Prove that φ is an isomorphism when $g = 1$ (hint: use covering space theory). It is surjective but not injective when $g > 1$. Can you construct an element of the kernel (called the *Torelli group*)?

3. Let Ω_n denote the set of bordism classes of smooth, oriented[1], closed n-dimensional manifolds.

   (a) Show that Ω_n is a group under the disjoint union operation (and identity given by the empty manifold).

   (b) Show that $\Omega_0 \cong \mathbb{Z}$.

   (c) Using the classification of 1 and 2-dimensional oriented manifolds, show that Ω_1 and Ω_2 are zero.

   (d) Let $\mathfrak{N}_n$ denote the corresponding bordism group of *unoriented* n-manifolds. Show that every element of $\mathfrak{N}_n$ is 2-torsion.

4. For any space X, write $X \cup \{\infty\}$ for the one-point compactification of X. Prove that there is an isomorphism

$$H^*_c(X) \cong H^*(X \cup \{\infty\}, \{\infty\}).$$

[1] An orientation of a connected 0-dimensional manifold M (i.e., a point), is a fundamental class: a choice of generator of $H_0(M) \cong \mathbb{Z}$.