# Homework #2 for MATH 8307: Algebraic Topology

April 7, 2015

**Due Date:** Wednesday 22 April in class.

1. Let $X$ and $Y$ be connected topological spaces.

   (a) If $X$ and $Y$ are CW complexes, recall that $X \times Y$ may be given the structure of a CW complex with a $(p+q)$-dimensional cell of the form $e^p \times e^q$ for each pair consisting of a $p$-cell $e^p$ in $X$ and $q$-cell $e^q$ in $Y$. Show that if $X$ and $Y$ each have a single 0-cell which we will take to be the basepoint, there is a cell structure on $X \wedge Y$ with a single 0-cell, and a $p+q$-dimensional cell for each pair consisting of a $p$-cell in $X$ and $q$-cell in $Y$, where neither $p$ nor $q$ is allowed to be 0.

   (b) Under the previous assumptions, if $X$ has no cells of dimension less than $n$, and $Y$ has no cells of dimension less than $m$ (other than the single 0-cell in each), show that $\pi_q(X \wedge Y) = 0$ for $q < m + n$.

   (c) Now let $X$ and $Y$ be arbitrary $(n-1)$-connected based topological spaces. Show, using some form of the homotopy excision theorem, that the homotopy groups of $X \vee Y$ vanish in degrees less than $n$, and that $\pi_n(X \vee Y) \cong \pi_n(X) \oplus \pi_n(Y)$. Don't use the Hurewicz theorem in your argument; this fact (for spheres) is required for the proof of Hurewicz.

2. Let $\mathbb{H}$ denote the division algebra of quaternions, and write $\mathbb{H}^\times = \mathbb{H} \setminus \{0\}$ for the group of units with the operation of multiplication. Define

$$\mathbb{H}P^n := (\mathbb{H}^{n+1} \setminus \{0\})/\mathbb{H}^\times$$

   where the action of $\mathbb{H}^\times$ on nonzero vectors in $\mathbb{H}^{n+1}$ is by:

$$\lambda \cdot (x_0, \ldots, x_n) = (\lambda x_0, \ldots, \lambda x_n)$$

   (a) Show that $\mathbb{H}P^n$ is homeomorphic to the quotient space $S^{4n+3}/S^3$, where $S^{4n+3} \subseteq \mathbb{H}^{n+1} \setminus \{0\}$ is the subset of norm 1, and $S^3 \leq \mathbb{H}^\times$ is the subgroup of norm 1.

   (b) Show that $\mathbb{H}P^1$ is homeomorphic to $S^4$.

   (c) Show that the quotient map $S^{4n+3} \to \mathbb{H}P^n$ is a principal $S^3$-fibre bundle.

   (d) Show that $\pi_k S^3 \cong \pi_{k+1} S^4$ when $k \leq 5$. Can you use the Freudenthal suspension theorem for this result?

   (e) Take as given Serre's theorem: $\pi_k S^{2n-1}$ is finite for $k \neq 2n-1$. Show that in contrast, $\pi_7 S^4$ contains an element of infinite order.

   (f) Let $G \leq \mathbb{H}^\times$ be a discrete subgroup, and define $X_n := (\mathbb{H}^{n+1} \setminus \{0\})/G$, where the action of $G$ is via $\mathbb{H}^\times$. Let $X = \cup_n X_n$ be the union of $X_n$ induced by the inclusions $\mathbb{H}^{n+1} \setminus \{0\} \subseteq \mathbb{H}^{n+2} \setminus \{0\}$. Compute the homotopy groups of $X$.