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String topology prospectra and Hochschild cohomology

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Abstract

We study string topology for classifying spaces of connected compact Lie groups, drawing connections with Hochschild cohomology and equivariant homotopy theory. First, for a compact Lie group G, we show that the string topology prospectrum LBG^{-TBG} is equivalent to the homotopy fixed-point prospectrum for the conjugation action of G on itself, $S^0[G]^{hG}$. Dually, we identify $LBG^{-\mathrm{ad}}$ with the homotopy orbit spectrum $(DG)_{hG}$, and study ring and co-ring structures on these spectra. Finally, we show that in homology, these products may be identified with the Gerstenhaber cup product in the Hochschild cohomology of $C^*(BG)$ and $C_*(G)$, respectively. These, in turn, are isomorphic via Koszul duality.

1. *Introduction*

Let G be a connected compact Lie group. The free loop space

$$LBG := \mathrm{Map}(S^1, BG)$$

of the classifying space of G is a natural object of study for topologists, representation theorists, and mathematical physicists. Its K-theory is related to an important example of a topological field theory, the Verlinde algebra of positive energy representations of the loop group LG [**11**]. In this article, we study LBG and natural field-theoretic algebraic structures which it supports from several points of view — string topology, Hochschild cohomology, and equivariant stable homotopy theory.

1.1. *Equivalences of* (*pro*)*spectra*

In string topology, one studies the free loop space LM of a closed, oriented, finite-dimensional manifold M. Using a combination of intersection theory on M and concatenation of loops with common basepoints, Chas and Sullivan [**4**] gave the shifted homology of LM the structure of a Gerstenhaber algebra. The ring structure was reinterpreted in the language of stable homotopy theory by Cohen and Jones in [**8**] in the form of a (Thom) ring spectrum LM^{-TM}.

Although BG is not a finite-dimensional manifold, it does admit a filtration by finite-dimensional manifolds. In [**15**], Salvatore and the first author defined an inverse system of ring spectra (or pro-ring spectrum) LBG^{-TBG} using this filtration and analogs of the string topology techniques of [**4**, **8**].

Consider the topological space G, equipped with the action of G by conjugation. In [**22**], the second author studied a ring spectrum $S^0[G]^{hG}$, the homotopy fixed-point spectrum for this action on the suspension spectrum $S^0[G]$. This spectrum is best understood as a pro-ring spectrum. We will employ the notation $S^0[G]^{hG}$ for the pro-ring spectrum, since we will almost always be working with it, and not the ring spectrum. One purpose of this paper is to show

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that there is an equivalence between the geometrically constructed LBG^{-TBG} and $S^0[G]^{hG}$, whose description is equivariant stable homotopy-theoretic.

THEOREM 1.1. *The transfer map τ^G defines an equivalence of pro-ring spectra*

$$LBG^{-TBG} \simeq S^0[G]^{hG}.$$

One should compare this result to [**17**], where Klein shows that for a Poincaré duality group G with classifying space $M = BG$ a Poincaré duality space of formal dimension d, there is an equivalence of the *spectrum* (not a prospectrum) of homotopy fixed points of the conjugation action of G on $S^0[G]$ with the string topology spectrum LM^{-TM}.

It is worth pointing out that the spectrum $S^0[G]^{hG}$ is equivalent to $THH^\bullet(S^0[G], S^0[G])$, the topological Hochschild cohomology of the suspension spectrum of G. This foreshadows Theorem 1.3 below.

In [**14**], the first author showed that the prospectrum LBG^{-TBG} is Spanier–Whitehead dual (in the sense of Christensen and Isaksen [**5**]) to a spectrum $LBG^{-\mathrm{ad}}$. There is a coproduct on that spectrum which, upon application of a cohomology theory, gives an (untwisted) analog of the Freed–Hopkins–Teleman product in twisted equivariant K-theory (or fusion product in the Verlinde algebra).

In light of this duality and Theorem 1.1, the following should be unsurprising.

THEOREM 1.2. *There is an equivalence of co-ring spectra $LBG^{-\mathrm{ad}} \simeq (DG)_{hG}$.*

Here $DG = F(S^0[G], S^0)$ is the Spanier–Whitehead dual of G, equipped with a naive G-action dual to the conjugation action on G. We describe the coproduct on the Borel construction $DG_{hG} = EG_+ \wedge_G DG$ in Section 3 below.

A remark on terminology is in order. Throughout this paper, the terms 'ring spectrum' and 'pro-ring spectrum' will be used to describe objects whose multiplication is associative up to homotopy. For more highly structured ring spectra, we will employ the S-algebras of [**9**]. Additionally, the term 'pro-ring spectrum' (respectively, 'pro-S-algebra') denotes an inverse system of ring spectra (respectively, S-algebras), rather than a monoid in the category of prospectra.

Further, we will not consider strict co-ring spectra, and only require them to be coassociative up to homotopy. Indeed, for most of this paper, we work in the homotopy category. However, the prospectrum $S^0[G]^{hG}$ is a (strict) pro-S-algebra, so Theorem 1.1 can be thought of as a rectification result for LBG^{-TBG}. This answers in the affirmative Conjecture 10 of [**14**].

1.2. *Homological computations*

A natural question is how to compute the (co)homology of these (pro)spectra. Let k be a field; all of our (co)chain and (co)homology groups will have coefficients in k.

Our approach is through Hochschild cohomology. For a differential-graded algebra A and a dg A-module M, $HH_*(A, M)$ and $HH^*(A, M)$ are the Hochschild homology and cohomology of A with coefficients in M. Recall that for any topological group K and topological space X, there are isomorphisms

$$H_*(LBK) \cong HH_*(C_*(K), C_*(K)) \quad \text{and} \quad H^*(LX) \cong HH_*(C^*(X), C^*(X)),$$

where $C_*(K)$ is given the structure of a dga via the Pontrjagin product, and $C^*(X)$ via the cup product of cochains.

In [**8**], Cohen–Jones modified the latter isomorphism to give an isomorphism of rings

$$H_*(LM^{-TM}) \cong HH^*(C^*(M), C^*(M))$$

for finite-dimensional manifolds M. We adapt both of these computations to the context of string topology on BG.

THEOREM 1.3. *If G is a connected compact Lie group, the following are mutually isomorphic:*

(1) *the ring $H_*^{\mathrm{pro}}(LBG^{-TBG})$, with the string topology product of* [**15**];
(2) *the ring $H^{-*}(LBG^{-\mathrm{ad}})$, with the ring structure induced by the 'fusion' coproduct on $LBG^{-\mathrm{ad}}$, defined in* [**14**];
(3) *the ring $HH^*(C^*(BG), C^*(BG))$, with the Gerstenhaber cup product;*
(4) *the ring $HH^*(C_*(G), C_*(G))$, with the Gerstenhaber cup product.*

In part (1), $H_*^{\mathrm{pro}}(LBG^{-TBG})$ denotes the inverse limit of the homologies of the terms in the prospectrum LBG^{-TBG}.

Here is a summary of the proof. To show the equivalence of parts (1) and (2), one uses the Spanier–Whitehead duality result of [**14**]. The isomorphism of the rings in parts (1) and (3) uses, as in [**8**], a cosimplicial model for LBG^{-TBG}. Finally, the differential-graded algebras $C^*(BG)$ and $C_*(G)$ are Koszul (or cobar) dual: $C^*(BG)$ is equivalent to the cobar complex for the differential-graded algebra $C_*(G)$ (and vice versa). As Hochschild cohomology is insensitive to Koszul duality [**10**, **16**], one obtains an isomorphism of the rings in parts (3) and (4).

Write this collection of isomorphisms in the following form:

$$\begin{array}{ccc} H_*^{\mathrm{pro}}(LBG^{-TBG}) & \longleftrightarrow & HH^*(C^*(BG), C^*(BG)) \\ {\scriptstyle D}\updownarrow & & \updownarrow{\scriptstyle K} \\ H^{-*}(LBG^{-\mathrm{ad}}) & \longleftrightarrow & HH^*(C_*(G), C_*(G)). \end{array}$$

In this diagram, the horizontal isomorphisms are 'geometric' in the sense that they come from explicit models for the spectra involved. The vertical isomorphism D is induced by Spanier–Whitehead duality, and K is induced by Koszul duality. Consequently, one may interpret this theorem as saying that the Spanier–Whitehead duality (of [**14**]) between the Chas–Sullivan and Freed–Hopkins–Teleman products is manifested in Hochschild cohomology as an aspect of Koszul duality.

Recent work of Vaintrob [**21**] gives an analog of the isomorphisms between parts (1) and (4) in the related case that $M^n = BG$ is a closed, oriented, aspherical manifold, and $G = \pi_1(M)$ is a discrete group. Namely, Vaintrob gives an isomorphism of BV algebras

$$HH^{n-*}(k[\pi_1(M)], k[\pi_1(M)]) \cong \Sigma^{-n} H_*(LM)$$

where $k[\pi_1(M)] \cong C_*(G; k)$ is the group algebra on the fundamental group of M.

Similar multiplicative structures coming from Chen–Ruan cohomology and string topology of orbifolds and stacks have been studied recently (see, for example, [**3**, **12**]). In the final part of this paper, we relate these constructions to the algebras described above.

2. *The pro-ring spectra*

Let us review the construction of these pro-ring spectra. Both will be defined using a filtration of EG — a contractible space upon which G acts freely — by finite-dimensional free G-manifolds. To do this, we proceed as follows. Because G is compact Lie, there exists a finite-dimensional, faithful representation V of G.

DEFINITION 2.1. Define E_nG to be the space of linear embeddings of V into $\mathbb{R}^n$.

Since the action of G on V is faithful, when E_nG is nonempty it is a free G-space, so fits into a principal G-fibration

$$G \longrightarrow E_nG \longrightarrow B_nG,$$

where we define $B_nG := E_nG/G$. Furthermore, by definition, E_nG and B_nG are both smooth manifolds. Finally, the filtered union of the sequence

$$E_1G \subseteq \cdots \subseteq E_nG \subseteq E_{n+1}G \subseteq \cdots$$

is the space of linear embeddings of V into R^∞ and contractible, so is therefore a model for EG; that is,

$$\operatorname{colim} E_nG = EG.$$

Similarly, $\operatorname{colim} B_nG = BG$.

EXAMPLE 2.2. For instance, when $G = SO(k)$, V may be taken to be $\mathbb{R}^k$, with the defining action of $SO(k)$ on V. Then B_nG is the Grassmannian of k-planes in $\mathbb{R}^n$, and E_nG is the corresponding Stiefel manifold.

2.1. *The string topology of BG*

DEFINITION 2.3. Let $\mathrm{Ad}(E_nG)$ denote the total space of the G-bundle

$$\pi : E_nG \times_G G \longrightarrow B_nG,$$

where G acts on itself by conjugation.

Since B_nG is a manifold, it has a tangent bundle, which one can pull back to $\mathrm{Ad}(E_nG)$ via π. In [**15**], it was shown that the Thom spectra

$$\mathrm{Ad}(E_nG)^{-TB_nG} := \mathrm{Ad}(E_nG)^{-\pi^*(TB_nG)}$$

are ring spectra, using a construction analogous to Cohen–Jones' construction of string topology operations in [**8**]. Specifically, one has a commutative diagram

$$\begin{array}{ccccc}
G \times G & \xleftarrow{=} & G \times G & \xrightarrow{\mu} & G \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ad}(E_nG) \times \mathrm{Ad}(E_nG) & \xleftarrow{\tilde{\Delta}} & \mathrm{Ad}(E_nG) \times_{B_nG} \mathrm{Ad}(E_nG) & \xrightarrow{\tilde{\mu}} & \mathrm{Ad}(E_nG) \\
\downarrow & & \downarrow & & \downarrow \\
B_nG \times B_nG & \xleftarrow{\Delta} & B_nG & \xrightarrow{=} & B_nG.
\end{array}$$

Because Δ is a finite codimension, so too is $\tilde{\Delta}$; hence both admit *umkehr* Pontrjagin–Thom collapse maps. Multiplication in the spectrum $\mathrm{Ad}(E_nG)^{-TB_nG}$ is given by the composite of the Pontrjagin–Thom collapse for $\tilde{\Delta}$ with $\tilde{\mu}$.

Furthermore, the natural inclusions $E_nG \subseteq E_{n+1}G$ define (via associated Pontrjagin–Thom maps) a tower of ring spectra

$$\mathrm{Ad}(E_1G)^{-TB_1G} \longleftarrow \cdots \longleftarrow \mathrm{Ad}(E_nG)^{-TB_nG} \longleftarrow \mathrm{Ad}(E_{n+1}G)^{-TB_{n+1}G} \longleftarrow \cdots.$$

Since there is a homotopy equivalence

$$LBG \simeq \mathrm{Ad}(EG),$$

this pro-ring spectrum is denoted LBG^{-TBG}.

2.2. *The naive homotopy fixed-point prospectrum*

Using the manifolds E_nG, one can define another pro-ring spectrum. Consider the function spectrum

$$F(E_nG_+, S^0[G])^G$$

of G-equivariant maps from E_nG to the suspension spectrum of G. Here $S^0[G]$ is regarded as a naive G-spectrum, with conjugation action. This may be given a ring product μ_n using the following diagram:

$$\begin{array}{ccc}
F(E_nG_+, S^0[G])^G \wedge F(E_nG_+, S^0[G])^G & \xrightarrow{smash} & F(E_nG \times E_nG_+, S^0[G \times G])^{G\times G} \\
 & & \big\downarrow i \\
\big\downarrow \mu_n & & F(E_nG \times E_nG_+, S^0[G \times G])^G \\
 & & \big\downarrow \Delta^* \\
F(E_nG_+, S^0[G])^G & \xleftarrow[\mu_*]{} & F(E_nG_+, S^0[G \times G])^G.
\end{array}$$

Here *smash* smashes two functions together. The spectrum

$$F(E_nG \times E_nG_+, S^0[G \times G])^G$$

is the space of maps that are equivariant with respect to the diagonal G action on each factor, so i is a forgetful map. The diagonal

$$\Delta : E_nG \longrightarrow E_nG \times E_nG$$

is a G-equivariant map, so induces Δ^*. Similarly, μ_* is induced by the multiplication $\mu : G \times G \to G$, which is a G-equivariant map (with respect to the diagonal action by conjugation).

It was shown in [**22**] that μ_n makes $F(E_nG_+, S^0[G])^G$ into an associative S-algebra (in fact, it is the first term of an operad in the stable category).

The natural inclusions $E_nG \subseteq E_{n+1}G$ are G-equivariant, so they induce maps of S-algebras

$$F(E_nG_+, S^0[G])^G \longleftarrow F(E_{n+1}G_+, S^0[G])^G,$$

which assemble into the pro-S-algebra

$$F(E_1G_+, S^0[G])^G \longleftarrow \cdots \longleftarrow F(E_nG_+, S^0[G])^G \longleftarrow F(E_{n+1}G_+, S^0[G])^G \longleftarrow \cdots .$$

For a naive G-spectrum X, the function spectrum $F(EG_+, X)^G$ is called the *homotopy fixed-point spectrum* X^{hG}. We will therefore denote this pro-S-algebra $S^0[G]^{hG}$.

2.3. *An alternate homotopy fixed-point prospectrum*

In equivariant stable homotopy theory there is another notion of suspension spectrum. For a space X, one may define the spectrum $S^0_G[X]$ whose nth space is

$$Q_G\Sigma^n X = \operatorname{colim}_V \Omega^V \Sigma^{V+\mathbb{R}^n} X,$$

and the colimit is taken over a complete G-universe of real finite-dimensional representations V of G. Here $S^V = V \cup \{\infty\}$ is the one-point compactification of V,

$$\Sigma^{V+\mathbb{R}^n} X = S^V \wedge S^n \wedge X$$

and $\Omega^V Y = F(S^V, Y)$ is the function space of based continuous maps $S^V \to Y$. We make $S^0_G[X]$ into a naive G-spectrum as follows: for $f \in \Omega^V \Sigma^{V+\mathbb{R}^n} X$, $g \in G$, and $\nu \in S^V = V \cup \{\infty\}$,

$$(g \cdot f)(\nu) = gf(\nu \cdot g^{-1}).$$

This extends over the colimit to give an action on each term of the spectrum. This, in turn, assembles into a naive action of G on $S^0_G[X]$.

Replacing $S^0[G]$ with $S^0_G[G]$ above (and using precisely the same arguments), we get a pro-S-algebra

$$F(E_1G_+, S^0_G[G])^G \longleftarrow \cdots \longleftarrow F(E_nG_+, S^0_G[G])^G \longleftarrow F(E_{n+1}G_+, S^0_G[G])^G \longleftarrow \cdots .$$

We will denote this pro-S-algebra by $S^0_G[G]^{hG}$.

There is a natural map

$$e : S^0[X] \longrightarrow S^0_G[X],$$

for one can regard the terms of $S^0[X]$ as a similar colimit, only taken over the family of trivial G-representation. This map is an equivariant map which is a nonequivariant equivalence [**1**, **13**], and thus gives an equivalence on homotopy fixed points. Consequently, the induced map of prospectra

$$e : S^0[G]^{hG} \longrightarrow S^0_G[G]^{hG}$$

is a pro-equivalence.

3. *The co-ring spectra*

In this section, we study the spectra $LBG^{-\mathrm{ad}}$ and $(DG)_{hG}$ and the coproducts defined on each.

3.1. *The spectrum $LBG^{-\mathrm{ad}}$*

We recall the definition of $LBG^{-\mathrm{ad}}$. Let $\mathfrak{g}$ be the Lie algebra of G, equipped with the adjoint action of G. Then one may form a flat bundle ad over $\mathrm{Ad}(EG) = EG \times_G G$ with total space

$$\mathrm{ad} := (EG \times G \times \mathfrak{g})/G.$$

The Thom spectrum of the virtual bundle $-\mathrm{ad}$ over $\mathrm{Ad}(EG) \simeq LBG$ is what we shall call $LBG^{-\mathrm{ad}}$.

3.2. *Group actions on variants of DG*

The group action of G on itself by conjugation induces a naive action of G on $DG = F(S^0[G], S^0)$ by preconjugation. We explore two variants on this action that are more geometrically defined.

The tangent bundle TG of G can be given the structure of a G-equivariant vector bundle, lifting the conjugation action on G: for $g \in G$, define $c_g : G \to G$ to be conjugation by g. For $h \in G$ and $\nu \in T_hG$, we define

$$g \cdot (h, \nu) := (c_g(h), d_h(c_g)(\nu)),$$

where $d_h(c_g)$ is the derivative of c_g at h. This construction makes the Thom spectrum G^{-TG} into a naive G-spectrum.

Alternatively, consider the Lie algebra $\mathfrak{g} := T_eG$ alone. It inherits an action of G as a subspace of TG; this is the adjoint action. This makes $S^{\mathfrak{g}} = \mathfrak{g} \cup \{\infty\}$ into a G-space, and thus $S^{-\mathfrak{g}}$ a naive G-spectrum. Smashing with the conjugation action on G gives a naive action of G on $S^{-\mathfrak{g}} \wedge G_+$.

Proposition 3.1. *There are equivariant equivalences*

$$DG \simeq G^{-TG} \simeq S^{-\mathfrak{g}} \wedge G_+.$$

Proof. The first equivalence is Atiyah duality. The second follows from the fact that G is parallelizable. □

Note that this gives an alternate construction of $LBG^{-\mathrm{ad}}$; from the construction of ad, it is apparent that

$$(S^{-\mathfrak{g}} \wedge G_+)_{hG} = LBG^{-\mathrm{ad}}.$$

Then Proposition 3.1 implies part of Theorem 1.2: taking homotopy orbits we see that

$$(DG)_{hG} \simeq (S^{-\mathfrak{g}} \wedge G_+)_{hG} = LBG^{-\mathrm{ad}}.$$

3.3. *Co-ring spectra*

In [**14**], it was shown that $\mathrm{Ad}(E)^{-\mathrm{ad}}$ is a co-ring spectrum, when $p : E \to M$ is a principal G-bundle over a finite-dimensional manifold M. It is not hard to extend this to the infinite-dimensional case $M = BG$.

PROPOSITION 3.2. *The spectrum $LBG^{-\mathrm{ad}} = (S^{-\mathfrak{g}} \wedge G_+)_{hG}$ is a homotopy coassociative co-ring spectrum.*

Proof. The multiplication map $m : G \times G \to G$ is a principal G-bundle; the fiber over the identity is $\{(g, g^{-1}), g \in G\} \cong G$. Consequently, there is a (stable) transfer map

$$m_! : S^{\mathfrak{g}} \wedge G_+ \longrightarrow G \times G_+$$

which is well defined up to homotopy. If we give $G \times G$ an action of G by conjugation in each factor, it is easy to see that m is equivariant. Therefore $m_!$ is also. Smashing with $S^{-2\mathfrak{g}}$ and taking homotopy orbits gives

$$M : (S^{-\mathfrak{g}} \wedge G_+)_{hG} \longrightarrow ((S^{-\mathfrak{g}} \wedge G_+) \wedge (S^{-\mathfrak{g}} \wedge G_+))_{hG}.$$

Here $M = (\mathrm{id}_{S^{-2\mathfrak{g}}} \wedge m_!)_{hG}$.

For any two naive G-spectra X and Y, there is a natural map

$$d : (X \wedge Y)_{hG} \longrightarrow X_{hG} \wedge Y_{hG}$$

induced by the diagonal on EG. We may define the coproduct on $LBG^{-\mathrm{ad}}$ to be the composite $d \circ M$.

To see that the coproduct is coassociative, first observe that $(m_! \wedge \mathrm{id}) \circ m_! = (\mathrm{id} \wedge m_!) \circ m_!$ as maps

$$S^{\mathfrak{g}} \wedge S^{\mathfrak{g}} \wedge G_+ \longrightarrow S^{\mathfrak{g}} \wedge G_+ \wedge G_+ \longrightarrow G_+ \wedge G_+ \wedge G_+,$$

since both are equal to the transfer map for the principal $G \times G$-bundle $G \times G \times G \to G$ given by three-term multiplication. Smashing with $S^{-3\mathfrak{g}}$ and taking homotopy orbits shows that the two compositions in the diagram below are equal.

$$\begin{array}{ccc}
 & & ((S^{-\mathfrak{g}} \wedge G_+) \wedge (S^{-\mathfrak{g}} \wedge G_+) \wedge (S^{-\mathfrak{g}} \wedge G_+))_{hG} \\
 & & \uparrow{\scriptstyle m_! \wedge \mathrm{id}} \\
(S^{-\mathfrak{g}} \wedge G_+)_{hG} & \xrightarrow{M} & ((S^{-\mathfrak{g}} \wedge G_+) \wedge (S^{-\mathfrak{g}} \wedge G_+))_{hG} \\
 & & \downarrow{\scriptstyle \mathrm{id} \wedge m_!} \\
 & & ((S^{-\mathfrak{g}} \wedge G_+) \wedge (S^{-\mathfrak{g}} \wedge G_+) \wedge (S^{-\mathfrak{g}} \wedge G_+))_{hG}.
\end{array}$$

Coassociativity then follows from the naturality of d and the coassociativity of the diagonal map on EG. □

Since G is a finite complex, the Spanier–Whitehead dual DG is also equipped with a natural co-ring spectrum structure, dual to the multiplication m in G. Since m is G-equivariant (with respect to the diagonal conjugation action), the coproduct on DG is also equivariant. This allows us to define a coproduct on the Borel construction $(DG)_{hG}$ by

$$(DG)_{hG} \xrightarrow{Dm} (DG \wedge DG)_{hG} \xrightarrow{d} (DG)_{hG} \wedge (DG)_{hG}.$$

It is evident that the Atiyah-duality equivalence $(DG)_{hG} \simeq (S^{-\mathfrak{g}} \wedge G_+)_{hG} = LBG^{-\mathrm{ad}}$ of Proposition 3.1 respects these co-ring structures. This completes the proof of Theorem 1.2.

4. *The proof of Theorem* 1.1

We begin the proof of Theorem 1.1 with the following lemma which asserts that the terms in each prospectrum are equivalent.

Lemma 4.1. *The transfer map for the principal G-fibration*

$$p : E_n G \times G \longrightarrow \mathrm{Ad}(E_n G)$$

gives rise to an equivalence

$$\tau_n : \mathrm{Ad}(E_n G)^{-TB_n G} \simeq F\big(E_n G_+, S^0_G[G]\big)^G.$$

Proof. Write $\mathfrak{g}$ for the Lie algebra of G and give it the adjoint G-action. Then one may form the vector bundle

$$(E_n G \times G \times \mathfrak{g})/G$$

over $\mathrm{Ad}(E_n G)$, with fiber $\mathfrak{g}$. We will write the Thom space of this bundle as $\mathrm{Ad}(E_n G)^{\mathfrak{g}}$.

Recall from [20] that the transfer map τ^G is an equivalence of spectra

$$\tau^G : \mathrm{Ad}(E_n G)^{\mathfrak{g}} \longrightarrow S^0_G[E_n G \times G]^G.$$

Let T denote the tangent bundle of $\mathrm{Ad}(E_n G)$, and $p^*(T)$ its pullback to $E_n G \times G$ via p. Then τ^G extends to an equivalence of Thom spectra

$$\tau_n = (\tau^G)^{-T} : \mathrm{Ad}(E_n G)^{\mathfrak{g}-T} \longrightarrow S^0_G[(E_n G \times G)^{-p^*(T)}]^G. \qquad (*)$$

There is a splitting of the tangent bundle of $\mathrm{Ad}(E_n G)$,

$$T = \mathfrak{g} \oplus \pi^*(TB_n G).$$

The base $B_n G$ embeds as the unit section of the projection π, and the vertical tangent bundle to π is $\mathfrak{g}$. Therefore the left-hand side of $(*)$ may be written as $\mathrm{Ad}(E_n G)^{-TB_n G}$.

Examine the right-hand side of $(*)$. Since p is a G-principal fibration, we know that

$$p^*(T) \oplus \mathfrak{g} = T(E_n G \times G),$$

and here $\mathfrak{g}$ is a trivial bundle over $E_n G \times G$. Note that $\mathfrak{g}$ is the lift of the tangent bundle of G to $E_n G \times G$. Therefore $p^*(T)$ is stably equivalent to the lift of $TE_n G$ to $E_n G \times G$. Therefore the right-hand side of $(*)$ may be written as

$$\big(E_n G^{-TE_n G} \wedge S^0_G[G]\big)^G.$$

Atiyah duality then tells us that, since $E_n G$ is a manifold, $E_n G^{-TE_n G}$ is the Spanier–Whitehead dual of $E_n G_+$,

$$E_n G^{-TE_n G} \simeq F(S^0[E_n G], S^0).$$

Using this along with the fact that for a finite spectrum X, there is an equivalence

$$F(Y, X) \simeq F(Y, S^0) \wedge X,$$

we see that the right-hand side of $(*)$ is

$$F\big(E_nG_+, S^0_G[G]\big)^G. \qquad \square$$

Lemma 4.2. *The maps τ_n are maps of ring spectra, up to homotopy.*

Proof. We will show that the following diagram homotopy commutes.

$$\begin{array}{ccc}
\mathrm{Ad}(E_nG)^{-TB_nG} \wedge \mathrm{Ad}(E_nG)^{-TB_nG} & \xrightarrow{\tau_n \wedge \tau_n} & \big(E_nG^{-TE_nG} \wedge S^0_G[G]\big)^G \wedge (E_nG^{-TE_nG} \wedge S^0_G[G])^G \\
\downarrow{\scriptstyle \tau_G^{G\times G}} & & \downarrow{\scriptstyle j\circ i\circ smash} \\
\mathrm{Ad}_G(E_nG \times E_nG)^{-(\mathfrak{g} \oplus T(B_nG \times B_nG))} & \xrightarrow{\tau_n'} & \big((E_nG \times E_nG)^{-T(E_nG \times E_nG)} \wedge S^0_G[G \times G]\big)^G \\
\downarrow{\scriptstyle \tau_{\Delta'}} & & \downarrow{\scriptstyle \Delta^*} \\
(\mathrm{Ad}(E_nG) \times_{B_nG} \mathrm{Ad}(E_nG))^{-TB_nG} & \xrightarrow{\tau_n''} & \big(E_nG^{-TE_nG} \wedge S^0_G[G \times G]\big)^G \\
\downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \mu_*} \\
\mathrm{Ad}(E_nG)^{-TB_nG} & \xrightarrow{\tau_n} & \big(E_nG^{-TE_nG} \wedge S^0_G[G]\big)^G.
\end{array}$$

Here

$$\mathrm{Ad}_G(E_nG \times E_nG) = (E_nG \times G \times E_nG \times G)/G,$$

where G acts diagonally. All of the horizontal maps are transfer maps: τ_n' is the transfer for the principal G-bundle

$$E_nG \times G \times E_nG \times G \longrightarrow (E_nG \times G \times E_nG \times G)/G,$$

Thomified with respect to the bundle $-(T \times T)$, and τ_n'' is the transfer for

$$E_nG \times G \times G \longrightarrow (E_nG \times G \times G)/G = \mathrm{Ad}(E_nG) \times_{B_nG} \mathrm{Ad}(E_nG),$$

Thomified with respect to $-(\mathfrak{g} \oplus \pi^*(TB_nG))$.

First, consider the top square. The map $\tau_G^{G\times G}$ is a transfer map similar to τ^G, arising from a Pontrjagin–Thom collapse map. Here it is Thomified with respect to $-(TB_nG \times TB_nG)$. The map j is induced by the natural map

$$S^0_G[G] \wedge S^0_G[G] \longrightarrow S^0_G[G \times G].$$

This top square commutes by the subgroup naturality of the transfer construction [20].

Next, consider the middle square. The map Δ^* is the Spanier–Whitehead dual of the diagonal $\Delta : E_nG \hookrightarrow E_nG \times E_nG$, hence is the Pontrjagin–Thom collapse map for the embedding Δ. Likewise, $\tau_{\Delta'}$ is the Pontrjagin–Thom collapse map for

$$\Delta' : (E_nG \times G \times G)/G \hookrightarrow (E_nG \times G \times E_nG \times G)/G.$$

The transfer maps are also the collapse maps, and the two ways around this square are the same collapse maps, up to homotopy.

In the third square, both vertical maps are induced by the group multiplication on G. Since this multiplication is equivariant for the diagonal conjugation action, the bottom square commutes [20].

Note that the composition $\tau_{\Delta'} \circ \tau_G^{G\times G}$ is the Pontrjagin–Thom collapse map for the embedding

$$\tilde{\Delta} : \mathrm{Ad}(E_nG) \times_{B_nG} \mathrm{Ad}(E_nG) \hookrightarrow \mathrm{Ad}(E_nG) \times \mathrm{Ad}(E_nG).$$

Thus, the composition $\mu_* \circ \tau_{\Delta'} \circ \tau_G^{G\times G}$ is the same as the ring spectrum multiplication on $\mathrm{Ad}(E_nG)^{-TB_nG}$ given in [**15**]. Furthermore, after identifying

$$\left(E_nG^{-TE_nG} \wedge S_G^0[G]\right)^G \simeq F\left(E_nG_+, S_G^0[G]\right)^G,$$

we see that the product given by $\mu_* \circ \Delta^* \circ j \circ i \circ \mathit{smash}$ is the same as that defined in Section 2.2. Thus τ_n is a map of ring spectra, up to homotopy. □

Lemma 4.3. *The maps τ_n commute with the maps defining the prospectra LBG^{-TBG} and $S^0[G]^{hG}$. That is, they define a map of prospectra.*

Proof. First observe that the structure maps

$$F\left(E_nG_+, S_G^0[G]\right)^G \longleftarrow F\left(E_{n+1}G_+, S_G^0[G]\right)^G$$

define maps

$$\left(E_nG^{-TE_nG} \wedge S_G^0[G]\right)^G \longleftarrow \left(E_{n+1}G^{-TE_{n+1}G} \wedge S_G^0[G]\right)^G,$$

which are induced by the Spanier–Whitehead dual of the inclusions $E_nG \subseteq E_{n+1}G$, and hence are the corresponding Pontrjagin–Thom collapse maps. We need to check that the diagram

$$\begin{array}{ccc} \mathrm{Ad}(E_nG)^{-TB_nG} & \xrightarrow{\tau_n} & \left(E_nG^{-TE_nG} \wedge S_G^0[G]\right)^G \\ \uparrow & & \uparrow \\ \mathrm{Ad}(E_{n+1}G)^{-TB_{n+1}G} & \xrightarrow{\tau_{n+1}} & \left(E_{n+1}G^{-TE_{n+1}G} \wedge S_G^0[G]\right)^G \end{array} \tag{1}$$

commutes. From the construction of the transfer map, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Ad}(E_nG)^{\mathfrak{g}\oplus\nu} & \xrightarrow{\tau_n} & \left(E_nG^{\nu} \wedge S_G^0[G]\right)^G \\ \uparrow & & \uparrow \\ \mathrm{Ad}(E_{n+1}G)^{\mathfrak{g}} & \xrightarrow{\tau_{n+1}} & (E_{n+1}G_+ \wedge S_G^0[G])^G, \end{array}$$

where the vertical maps are the collapse maps and ν is the pullback of the normal bundle of B_nG in $B_{n+1}G$. Thomifying the diagram above with respect to $-T(\mathrm{Ad}(E_{n+1}G))$ yields diagram (1). □

Theorem 1.1 follows from these three lemmata; the maps τ_n assemble into an equivalence of pro-ring spectra. It is worth pointing out that these methods extend to give an equivalence $\mathrm{Ad}(M \times_G EG)^{-TBG} \simeq S^0[M]^{hG}$ of pro-ring spectra for any G-monoid M.

5. *Hochschild cohomology of $C^*(BG)$*

The purpose of this section is to prove the equivalence of parts (1) and (3) in Theorem 1.3. We begin with a cosimplicial description of the terms in the prospectrum LBG^{-TBG}. We use this to give an intermediate result describing the homology of these terms. This is then assembled into the result using various limit arguments.

Because we have assumed that G is connected, BG is simply connected, and for n sufficiently large, so too is B_nG. This ensures that the spectral sequences that we employ will converge.

5.1. *A cosimplicial model for* $\mathrm{Ad}(E_nG)^{-TB_nG}$

In this section, we construct a cosimplicial ring spectrum $\mathrm{Ad}_n^\bullet$ with the property that

$$\mathrm{Tot}(\mathrm{Ad}_n^\bullet) \simeq \mathrm{Ad}(E_nG)^{-TB_nG}.$$

The bulk of this section is adapted directly from [**6**, **8**], so we will be brief except in instances where our construction differs substantially.

One can realize the free loop space of BG as the totalization of the cosimplicial space $\mathrm{Map}(S^1_\bullet, BG)$,

$$LBG = \mathrm{Map}(S^1, BG) = \mathrm{Map}(|S^1_\bullet|, BG) = \mathrm{Tot}(\mathrm{Map}(S^1_\bullet, BG)).$$

Here $S^1_\bullet$ is the simplicial set whose geometric realization is the circle; $S^1_\bullet$ has $k+1$ k-dimensional simplices. Hence

$$\mathrm{Map}\big(S^1_k, BG\big) = BG^{\times k+1}.$$

The cofaces and codegeneracies are given by various diagonals and projections.

Proposition 5.1. *The space* $\mathrm{Ad}(E_nG)$ *is homotopy-equivalent to the totalization of the subcosimplicial space of* $\mathrm{Map}(S^1_\bullet, BG)$ *whose* kth *space is*

$$B_nG \times BG^{\times k}.$$

Proof. The subcosimplicial space described is carried via the equivalence

$$\mathrm{Tot}(\mathrm{Map}(S^1_\bullet, BG)) = LBG$$

homeomorphically to the subspace $L_nBG \subseteq LBG$ given by those loops whose basepoint lies in $B_nG \subseteq BG$.

Recall that $\mathrm{Ad}(E_nG) = E_nG \times_G G$; as such $\mathrm{Ad}(E_nG)$ is the fiber product

$$\begin{array}{ccc} \mathrm{Ad}(E_nG) & \xrightarrow{\subseteq} & \mathrm{Ad}(EG) \\ \downarrow & & \downarrow \\ B_nG & \xrightarrow{\subseteq} & BG. \end{array}$$

Similarly, L_nBG is the fiber product

$$\begin{array}{ccc} L_nBG & \xrightarrow{\subseteq} & LBG \\ \downarrow & & \downarrow \\ B_nG & \xrightarrow{\subseteq} & BG. \end{array}$$

Since the fibrations $LBG \to BG$ and $\mathrm{Ad}(EG) \to BG$ are equivalent, these fiber squares imply that L_nBG and $\mathrm{Ad}(E_nG)$ are equivalent. □

We now desuspend this construction by the tangent bundle of B_nG. For this we need the following construction. Consider the composite map

$$B_nG \xrightarrow{\Delta} B_nG \times B_nG \xrightarrow{1\times i} B_nG \times BG.$$

This is the 0th coface of the cosimplicial space which totalizes to L_nBG. The pullback of $TB_nG \times 0$ to B_nG via this map is once again TB_nG. Thus, we have an induced map

$$\mu_R : B_nG^{-TB_nG} \longrightarrow B_nG^{-TB_nG} \wedge BG_+.$$

Making the same construction with $1 \times i$ replaced by $i \times 1$ defines a similar map

$$\mu_L : B_n G^{-TB_n G} \longrightarrow BG_+ \wedge B_n G^{-TB_n G}.$$

We describe these maps by the element-theoretic formulae

$$\mu_L(u) = (y_L, \nu_L), \quad \mu_R(u) = (\nu_R, y_R).$$

Though this does not quite make sense as spectra do not have elements, we hope the meaning is clear.

DEFINITION 5.2. For a group G and an integer $n > 0$, define a cosimplicial spectrum $\mathrm{Ad}_n^\bullet$ whose kth term is

$$\mathrm{Ad}_n^k := B_n G^{-TB_n G} \wedge BG_+^{\times k},$$

with coface and codegeneracy maps defined by the element-theoretic formulae

$$\begin{aligned}
\delta_0(u; x_1, \ldots, x_{k-1}) &= (\nu_R; y_R, x_1, \ldots, x_{k-1}),\\
\delta_i(u; x_1, \ldots, x_{k-1}) &= (u; x_1, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_{k-1}),\\
&\quad 1 \leqslant i \leqslant k-1\\
\delta_k(u; x_1, \ldots, x_{k-1}) &= (\nu_L; x_1, \ldots, x_{k-1}, y_L),\\
\sigma_i(u; x_1, \ldots, x_{k+1}) &= (u; x_1, \ldots, x_i, x_{i+2}, \ldots, x_{k+1})\\
&\quad 0 \leqslant i \leqslant k.
\end{aligned}$$

Define a map

$$m_{k,l} : \mathrm{Ad}_n^k \wedge \mathrm{Ad}_n^l \longrightarrow \mathrm{Ad}_n$$

by the composite

$$\begin{array}{ccc}
\left(B_n G^{-TB_n G} \wedge BG_+^{\times k}\right) \wedge \left(B_n G^{-TB_n G} \wedge BG_+^{\times l}\right) & \xrightarrow{T} & B_n G^{-TB_n G} \wedge B_n G^{-TB_n G} \wedge BG_+^{\times k+l} \\
& \searrow^{m_{k,l}} & \downarrow{\scriptstyle m \wedge 1} \\
& & B_n G^{-TB_n G} \wedge BG_+^{\times k+l},
\end{array}$$

where T switches factors, and m is multiplication in the ring spectrum $B_n G^{-TB_n G}$.

After totalization, the maps $m_{k,l}$ define a multiplication

$$\mathrm{Tot}(\mathrm{Ad}_n^\bullet) \wedge \mathrm{Tot}(\mathrm{Ad}_n^\bullet) \longrightarrow \mathrm{Tot}(\mathrm{Ad}_n^\bullet)$$

which makes $\mathrm{Tot}(\mathrm{Ad}_n^\bullet)$ into a ring spectrum.

THEOREM 5.3. *There is an equivalence of ring spectra*

$$\mathrm{Ad}(E_n G)^{-TB_n G} \xrightarrow{\simeq} \mathrm{Tot}(\mathrm{Ad}_n^\bullet).$$

Proof. The equivalence of these spectra follows from Proposition 5.1. The proof that the equivalence preserves ring multiplication is identical to the proof in [8] that LM^{-TM} and $\mathrm{Tot}(\mathbb{L}_M^\bullet)$ are equivalent ring spectra. □

5.2. *The homology of* $\mathrm{Ad}(E_n G)^{-TB_n G}$

Recall that for any space X, the singular cochain complex, $C^*(X)$, is a differential-graded algebra via the cup product of cochains. Using left and right multiplication, $C^*(X)$ becomes a $C^*(X)$-differential-graded bimodule. Maps of spaces induce maps of differential-graded algebras, so the maps

$$B_nG \xrightarrow{i_n} B_{n+1}G \xrightarrow{i} BG$$

make $C^*(B_nG)$, $C^*(B_{n+1}G)$ and $C^*(BG)$ into $C^*(BG)$-bimodule algebras. Further, the maps i_n^* and i^* are maps of bimodule algebras. One may therefore form the Hochschild cohomology

$$HH^*(C^*(BG), C^*(B_nG))$$

which becomes a ring under the cup product of Hochschild cochains. This allows us to describe the homology of individual terms of the prospectrum LBG^{-TBG}.

Theorem 5.4. *There is a ring isomorphism*

$$HH^*(C^*(BG), C^*(B_nG)) \cong H_*(\mathrm{Ad}(E_nG)^{-TB_nG}).$$

Proof. Theorem 5.3 gives the following equivalence of chain complexes:

$$C_*(\mathrm{Ad}(E_nG)^{-TB_nG}) \simeq \mathrm{Tot}\big(C_*\big(BG_+^{\times\bullet} \wedge B_nG^{-TB_nG}\big)\big).$$

Using the Eilenberg–Zilber theorem and the Atiyah duality, the right-hand side is equivalent to the totalization of the cosimplicial chain complex

$$k \mapsto C_*(BG)^{\otimes k} \otimes C^*(B_nG).$$

Define a chain map $g_k : C_*(BG)^{\otimes k} \otimes C^*(B_nG) \to \mathrm{Hom}(C^*(BG)^{\otimes k}, C^*(B_nG))$ by adjunction and evaluation,

$$e_1 \otimes \cdots \otimes e_k \otimes f \longmapsto ((f_1 \otimes \cdots \otimes f_k) \mapsto f_1(e_1) \cdots f_k(e_k) \cdot f).$$

It is easy to verify that the collection $\{g_k, k \geqslant 0\}$ defines a cosimplicial map

$$g : C_*(BG)^{\otimes\bullet} \otimes C^*(B_nG) \longrightarrow CH^\bullet(C^*(BG), C^*(B_nG)).$$

The theorem follows if we show that g induces a homology isomorphism upon totalization.

To see this, we notice that there are spectral sequences that compute the homology of the two terms in question,

$$\begin{aligned} E_1 := H_*(BG)^{\otimes\bullet} \otimes H^*(B_nG) \quad &\Longrightarrow \quad H_*(\mathrm{Tot}(C_*(BG)^{\otimes\bullet} \otimes C^*(B_nG))) \\ &\text{and} \\ E_1' := CH^*(H^*(BG), H^*(B_nG)) \quad &\Longrightarrow \quad HH^*(C^*(BG), C^*(B_nG)). \end{aligned}$$

The cosimplicial chain map g induces a map g_* between the spectral sequences; we claim that $g_* : E_1 \to E_1'$ is an isomorphism. In each cosimplicial degree k, the map

$$g_* : H_*(BG)^{\otimes k} \otimes H^*(B_nG) \longrightarrow \mathrm{Hom}(H^*(BG)^{\otimes k}, H^*(B_nG))$$

is a graded isomorphism because $H^*(B_nG)$ is finite-dimensional, and $H_*(BG)^{\otimes k}$ is finite-dimensional in each degree. Consequently, g_* is an isomorphism of spectral sequences; hence g induces an isomorphism in homology after totalization.

The cosimplicial product structure on $\mathrm{Ad}_n^\bullet$ is seen immediately to coincide with the cup product of Hochschild cochains. Consequently, this is an isomorphism of rings. □

5.3. *Limit arguments*

Examine the direct system

$$B_1G \longrightarrow \cdots \longrightarrow B_nG \longrightarrow B_{n+1}G \longrightarrow \cdots \longrightarrow BG.$$

Applying the (integral) singular chain and cochain complex functors produces direct and inverse systems of chain (respectively, cochain) complexes. Since BG is given the weak (or limit) topology of the system, this allows us to identify the singular chain complex of BG,

$$C_*(BG) = \varinjlim C_*(B_nG).$$

Standard properties of limits and colimits then imply that

$$C^*(BG) = \varprojlim C^*(B_nG).$$

PROPOSITION 5.5. *There is an isomorphism of cochain complexes*

$$CH^*(C^*(BG), C^*(BG)) \cong \varprojlim CH^*(C^*(BG), C^*(B_nG)).$$

Proof. For a given differential-graded algebra A, the Hochschild cochain functor $CH^*(A, \cdot)$ is covariant in the module variable for chain maps of differential-graded modules over A.

Recall that

$$i_n^* : C^*(B_{n+1}G) \longrightarrow C^*(B_nG)$$

is a chain map and a map of $C^*(BG)$-modules. Consequently, the map induced by i_n^*

$$CH^*(C^*(BG), C^*(B_{n+1}G)) \longrightarrow CH^*(C^*(BG), C^*(B_nG))$$

is a chain map. Therefore $\varprojlim CH^*(C^*(BG), C^*(B_nG))$ is also a chain complex.

We also know that

$$i^* : C^*(BG) \longrightarrow C^*(B_nG)$$

is a chain map and a map of $C^*(BG)$-modules. So the maps

$$CH^*(C^*(BG), C^*(BG)) \longrightarrow CH^*(C^*(BG), C^*(B_nG))$$

are chain maps. Since they are coherent across the inverse system, they assemble into a chain map

$$CH^*(C^*(BG), C^*(BG)) \longrightarrow \varprojlim CH^*(C^*(BG), C^*(B_nG)).$$

Generally, if Z is an Abelian group and

$$X_0 \longleftarrow X_1 \longleftarrow \cdots$$

an inverse system of Abelian groups, there is a canonical isomorphism (of groups)

$$\mathrm{Hom}(Z, \varprojlim X_i) \cong \varprojlim \mathrm{Hom}(Z, X_i).$$

Consequently, the map induced by i^* is an isomorphism

$$\begin{aligned} CH^k(C^*(BG), C^*(BG)) &\cong CH^k(C^*(BG), \varprojlim C^*(B_nG)) \\ &\cong \varprojlim CH^k(C^*(BG), C^*(B_nG)) \end{aligned}$$

for each k. The previous comments imply that this isomorphism is one of the chain complexes. □

Using a $\varprojlim^1$ argument and some topology, we may conclude the following homological analog.

COROLLARY 5.6. *There is an isomorphism of rings*

$$HH^*(C^*(BG), C^*(BG)) \cong \varprojlim HH^*(C^*(BG), C^*(B_nG)).$$

Proof. The tower

$$\cdots \longleftarrow C^*(B_nG) \longleftarrow C^*(B_{n+1}G) \longleftarrow \cdots$$

satisfies the Mittag–Leffler condition; consequently, so does the tower

$$\cdots \longleftarrow CH^*(C^*(BG), C^*(B_nG)) \longleftarrow CH^*(C^*(BG), C^*(B_{n+1}G)) \longleftarrow \cdots .$$

Using this fact and the previous proposition, we see that there is a short exact sequence

$$0 \longrightarrow \varprojlim{}^1 HH^*(C^*(BG), C^*(B_nG)) \longrightarrow HH^*(C^*(BG), C^*(BG)) \longrightarrow \varprojlim HH^*(C^*(BG), C^*(B_nG)) \longrightarrow 0.$$

Recall that we have shown that

$$HH^*(C^*(BG), C^*(B_nG)) \cong H_*(\mathrm{Ad}(E_nG)^{-TB_nG}).$$

So the $\varprojlim^1$ term vanishes if we can show that maps

$$\mathrm{Ad}(E_nG)^{-TB_nG} \longleftarrow \mathrm{Ad}(E_{n+1}G)^{-TB_{n+1}G}$$

satisfy the Mittag–Leffler condition in homology. Since the Spanier–Whitehead dual of $\mathrm{Ad}(E_nG)^{-TB_nG}$ is $\mathrm{Ad}(E_nG)^{-\mathrm{ad}}$, this is equivalent to showing that the inclusions

$$\mathrm{Ad}(E_nG) \longrightarrow \mathrm{Ad}(E_{n+1}G) \qquad (*)$$

satisfy the Mittag–Leffler condition in cohomology. By construction, the connectivity of the inclusions $B_nG \hookrightarrow BG$ increases with n; hence the same is true for the inclusions $\mathrm{Ad}(E_nG) \hookrightarrow \mathrm{Ad}(EG)$. This implies that $(*)$ does, in fact, satisfy the Mittag–Leffler condition in cohomology.

Since each map in the tower of coefficients is a ring homomorphism (in fact, a $C^*(BG)$-bimodule algebra map), the resulting isomorphism is one of rings. □

5.4. *A proof of* (1) ⇔ (3) *in Theorem* 1.3

Recall that we define

$$H_*^{\mathrm{pro}}(LBG^{-TBG}) := \varprojlim H_*(\mathrm{Ad}(E_nG)^{-TB_nG}).$$

Using Theorem 5.4 and Corollary 5.6, we therefore have

$$H_*^{\mathrm{pro}}(LBG^{-TBG}) \cong \varprojlim HH^*(C^*(BG), C^*(B_nG)) \cong HH^*(C^*(BG), C^*(BG)).$$

The ring structure on the left-hand side is defined to be the inverse limit of the ring structures on $H_*(\mathrm{Ad}(E_nG)^{-TB_nG})$. We have just shown the same to be true for the right-hand side; hence this isomorphism is one of rings.

6. *Spanier–Whitehead duality*

In this section, we show the isomorphism between the rings in parts (1) and (2) of Theorem 1.3.

Since $LBG \simeq \mathrm{Ad}(EG) = \varinjlim \mathrm{Ad}(E_nG)$, there is an exact sequence

$$0 \longrightarrow \varprojlim{}^1 H^*(\mathrm{Ad}(E_nG)^{-\mathrm{ad}}) \longrightarrow H^*(LBG^{-\mathrm{ad}}) \longrightarrow \varprojlim H^*(\mathrm{Ad}(E_nG)^{-\mathrm{ad}}) \longrightarrow 0.$$

Using the same arguments as in Section 5.3, we see that the $\varprojlim^1$ term vanishes.

In [**14**], the first author has shown that the spectra $\mathrm{Ad}(E_nG)^{-TB_nG}$ and $\mathrm{Ad}(E_nG)^{-\mathrm{ad}}$ are Spanier–Whitehead dual. Since these are finite spectra, we may conclude that

$$H_*(\mathrm{Ad}(E_nG)^{-TB_nG}) \cong H^{-*}(\mathrm{Ad}(E_nG)^{-\mathrm{ad}}).$$

Moreover, since Spanier–Whitehead duality carries the product on $\mathrm{Ad}(E_nG)^{-TB_nG}$ to the coproduct on $\mathrm{Ad}(E_nG)^{-\mathrm{ad}}$, this isomorphism is one of rings. Therefore there is a ring isomorphism

$$H_*^{\mathrm{pro}}(LBG^{-TBG}) := \varprojlim H_*(\mathrm{Ad}(E_nG)^{-TB_nG}) \cong \varprojlim H^{-*}(\mathrm{Ad}(E_nG)^{-\mathrm{ad}}) \cong H^{-*}(LBG^{-\mathrm{ad}}).$$

7. *Hochschild cohomology of $C_*(G)$ and Koszul duality*

7.1. *The bar and cobar constructions*

We recall the bar and cobar constructions for differential-graded (co)algebras. To begin with, let R be a connected, augmented, associative dga over a field k. Recall that for a right R-module M, and a left R-module N, the *two-sided bar construction* $B(M,R,N)$ is the realization of the simplicial chain complex $B_\bullet(M,R,N)$, given by

$$B_n(M,R,N) = M \otimes R^{\otimes n} \otimes N, \quad n \in \mathbb{N}$$

whose faces are given by multiplication in R and the module structure on M and N (and degeneracies are given by insertion of a unit). Recall, further, that $B(R) := B(k,R,k)$, the classic bar construction on R, is a differential-graded coalgebra, and $B(M,R,k)$ and $B(k,R,N)$ are, respectively, the right and the left comodules for $B(R)$.

Dually, for a supplemented, coassociative coalgebra S and the right and left comodules P and Q for S, the *two-sided cobar construction* $\Omega(P,S,Q)$ is the totalization of the cosimplicial chain complex

$$\Omega^n(P,S,Q) = P \otimes S^{\otimes n} \otimes Q, \quad n \in \mathbb{N}$$

whose cofaces are given by comultiplication in S and the comodule structure on P and Q, and whose codegeneracies come from the counit in S. Write $\Omega(S) := \Omega(k,S,k)$; this is a differential-graded algebra.

A relationship between these two constructions is as follows. Let S be a differential-graded coalgebra over a field k which is finite-dimensional in each degree. Then the dual $S^\vee = \mathrm{Hom}(S,k)$ is a differential-graded algebra, and there is an isomorphism of differential-graded coalgebras

$$B(S^\vee) \cong (\Omega(S))^\vee. \qquad (*)$$

7.2. *Koszul duality*

To our knowledge, there are at least two approaches to proving that Hochschild cohomology is insensitive to Koszul duality, using [**10**] and [**16**]. We recall these results.

A supplemented coalgebra $S = \overline{S} \oplus k$ is said to be *conilpotent* if, for every $x \in \overline{S}$, there is an n so that the nth iterated reduced comultiplication vanishes on x. In [**10**], Felix, Menichi, and Thomas proved that if S is locally conilpotent, non-negatively graded, and finitely generated in each degree, then there is an isomorphism of Gerstenhaber algebras

$$HH^*(\Omega S, \Omega S) \cong HH^*(S^\vee, S^\vee).$$

This was realized via a chain map

$$CH^*(\overline{\Omega} S, \overline{\Omega} S) \longrightarrow CH^*(S^\vee, S^\vee).$$

Here $\overline{\Omega} S$ is the reduced cobar construction, which is equivalent to ΩS.

Dually, let R be a differential-graded algebra, and write $R^!$ for the Koszul dual dga of R. That is, $R^!$ is the linear dual of $B(R)$,

$$R^! = (B(R))^\vee = \mathrm{Hom}(B(R), k).$$

In [**16**], Hu gave a proof that there is an equivalence of chain complexes

$$CH^*(R,R) \simeq CH^*(R^!, R^!),$$

assuming that $H_*(R^!)$ is a finite-dimensional k-vector space. Though not explicitly stated, it is does follow from the proof given there that this induces a ring isomorphism in Hochschild cohomology (we include a sketch below). These two results are clearly related via the isomorphism $(*)$.

Proposition 7.1. *The equivalence* $CH^*(R,R) \simeq CH^*(R^!,R^!)$ *of* [16] *induces a ring isomorphism*

$$HH^*(R,R) \cong HH^*(R^!,R^!).$$

Proof. We summarize the essential points of the proof given in [16] in order to show that this isomorphism is one of rings. Hu considers the bicosimplicial object

$$X^{\bullet,\bullet} := \mathrm{Hom}_{R\otimes R^{\mathrm{op}}}(B_\bullet(R,R,R), \Omega^\bullet(B(R,R,k), B(k,R,k), B(k,R,R))).$$

Recall that if R is connected, there is an equivalence $R \to \Omega(B(R))$ of differential-graded algebras. Further, there are R-module equivalences $B(R,R,k) \to k \leftarrow B(k,R,R)$, so we have an equivalence of $R\otimes R^{\mathrm{op}}$-modules

$$R \longrightarrow \Omega(B(R)) \longleftarrow \Omega^\bullet(B(R,R,k), B(k,R,k), B(k,R,R))).$$

Furthermore, $B_\bullet(R,R,R) \to R$ is a free $R\otimes R^{\mathrm{op}}$-resolution (over $B_\bullet(k,R,k)$). Therefore $X^{\bullet,\bullet}$ computes the Hochschild cohomology of R,

$$HH^*(R,R) = \mathrm{RHom}_{R\otimes R^{\mathrm{op}}}(R,R) = H_*(X^{\bullet,\bullet}).$$

Moreover, by $R\otimes R^{\mathrm{op}}$-freeness, there is an isomorphism

$$\begin{aligned} X^{\bullet,\bullet} &= \mathrm{Hom}_k(B_\bullet(k,R,k), \Omega^\bullet(B(R,R,k), B(k,R,k), B(k,R,R))) \\ &= \mathrm{Hom}_k(B_\bullet(R), \Omega^\bullet(B(R,R,k), B(R), B(k,R,R))). \end{aligned}$$

Using the equivalences $B(R,R,k) \simeq k \simeq B(k,R,R)$, we see that this complex is equivalent to

$$\mathrm{Hom}_k(B_\bullet(R), \Omega^\bullet(k, B(R), k)),$$

which is, in turn, isomorphic to

$$\mathrm{Hom}_{B(R)\otimes B(R)^{\mathrm{op}}-\mathrm{comod}}(B_\bullet(R), \Omega^\bullet(B(R), B(R), B(R))), \qquad (**)$$

since $\Omega^\bullet(B(R), B(R), B(R)))$ is a cofree $B(R)\otimes B(R)^{\mathrm{op}}$-comodule on $\Omega^\bullet(k, B(R), k)$. Using the homological finiteness of $R^!$, we notice that $(**)$ computes

$$HH^*(R^!,R^!) = \mathrm{RHom}_{R^!\otimes R^{!\mathrm{op}}}(R^!,R^!) = \mathrm{RHom}_{B(R)\otimes B(R)^{\mathrm{op}}-\mathrm{comod}}(B(R), B(R))$$

because $\Omega^\bullet(B(R), B(R), B(R))$ is a cofree resolution of $B(R)$ in the category of $B(R)\otimes B(R)^{\mathrm{op}}$-comodules.

Now, the Gerstenhaber cup product in $HH^*(A,A)$ can be identified with the Yoneda (composition) product in $\mathrm{RHom}_{A\otimes A^{\mathrm{op}}}(A,A)$. The isomorphism

$$HH^*(R,R) = H_*(X^{\bullet,\bullet}) \cong HH^*(R^!,R^!)$$

given above comes from the quasi-isomorphism

$$X^{\bullet,\bullet} \simeq \mathrm{Hom}_{B(R)\otimes B(R)^{\mathrm{op}}-\mathrm{comod}}(B_\bullet(R), \Omega^\bullet(B(R), B(R), B(R)))$$

which preserves the composition in each of these Hom-complexes. ☐

7.3. *Application to* $C^*(BG)$

We will apply these results on the case at hand, using the coalgebra $S = C_*(BG)$ or dually $R = S^\vee = C^*(BG)$.

It is well known (using the Eilenberg–Moore spectral sequence, for instance) that there is a homotopy equivalence of the dga

$$C_*(G) \simeq C_*(\Omega BG) \simeq \Omega(C_*(BG)) \simeq (C^*(BG))^!,$$

and our assumption that G is compact Lie ensures that the homology of the Koszul dual is finite. Using Hu's theorem, we conclude that there is a ring isomorphism

$$HH^*(C^*(BG), C^*(BG)) \cong HH^*(C_*(G), C_*(G)).$$

It is unclear whether we may use [10] to give an alternate proof and strengthen this isomorphism to one of the Gerstenhaber algebras. For if we use the singular cochain complex, $S = C_*(BG)$ is far from finite-dimensional in each degree. It may be possible to construct a quasi-isomorphic coalgebra $S' \simeq S$ which satisfies the assumptions of Felix–Menichi–Thomas' theorem (another of their results implies that the Gerstenhaber structure of Hochschild cohomology is preserved by quasi-isomorphism of the dga). The simple connectivity of BG and local finiteness of $H_*(BG)$ suggest that one may be able to find a locally finite simplicial set $Y_\bullet$ whose geometric realization is homotopy-equivalent to BG. Then S' could be taken to be the simplicial chain complex of $Y_\bullet$. But we do not know a construction of such a simplicial set $Y_\bullet$.

8. *Relationship to string topology constructions*

We have already seen that several of our results have interpretations in terms of string topology: in Theorem 1.1, LBG^{-TBG} arises from the string topology of BG, and the results of Section 5 are analogs of the Cohen–Jones string topology theorem that for a simply connected manifold M, $\mathbb{H}_*(LM) \cong H^*(C^*(M), C^*(M))$ as graded algebras [8]. In this section, we will give an interpretation of the co-ring spectrum $LBG^{-\mathrm{ad}}$ in terms of string topology, using string topology constructions for stacks.

In [7], Cohen and Godin defined a non-counital Frobenius algebra structure on $h_*(LM)$, with multiplication given by the Chas–Sullivan product. In [18], Lupercio, Uribe and Xicoténcatl extended the Chas–Sullivan construction to loop orbifolds. Using this, a localization principle allowed them in [19] to define an associative multiplication on $H_*(\Lambda[X^n/\Sigma_n])$, the homology of the inertia orbifold of a symmetric product. They then showed that this multiplication is Poincaré dual to a *virtual intersection product* on $H^*(\Lambda[X^n/\Sigma_n])$, which, with coauthors González and Segovia in [12], was identified with $H^*_{CR}(T^*[X^n/\Sigma_n])$, the Chen–Ruan cohomology of the cotangent bundle of $[X^n/\Sigma_n]$. This product is part of a Frobenius algebra structure in the Chen–Ruan cohomology.

Behrend, Ginot, Noohi, and Xu (BGNX) gave similar constructions in [2, 3], where they define a Frobenius algebra structure on $H_*(\Lambda\mathfrak{X})$, the homology of the inertia stack of an oriented differentiable stack $\mathfrak{X}$. Unlike the Frobenius algebra in the Chen–Ruan cohomology, this structure is not necessarily unital nor counital. In this structure, the multiplication is given by a stacky version of the Chas–Sullivan product, and the coproduct is given by a stacky version of the Cohen–Godin coproduct.

In the case that $\mathfrak{X} = [*/G]$, the classifying stack of a compact Lie group G, the inertia stack $\Lambda\mathfrak{X}$ is the quotient stack $[G/G]$ where G acts on itself by conjugation. Then

$$H_*(\Lambda\mathfrak{X}) = H_*([G/G]) = H_*(\mathrm{Ad}(EG)) \cong H_*(LBG),$$

so it is natural to ask whether the 'inertia Frobenius algebra' studied in [2, 12] is related to the co-ring spectrum $LBG^{-\mathrm{ad}}$. The relationship is clearest when we consider instead the Frobenius algebra structure on $H^*(\Lambda[*/G])$, induced via the universal coefficient theorem as in [2]. The following theorem says that the product (defined by BGNX) on $H^*([G/G])$, and hence the coproduct on $H_*([G/G])$, are induced by the coproduct on the $LBG^{-\mathrm{ad}}$ from Proposition 3.2.

Proposition 8.1. *The product on the inertia Frobenius algebra $H^*([G/G])$ is equal to the product on $H^*(LBG)$ induced from the co-ring spectrum structure on $LBG^{-\mathrm{ad}}$.*

Proof. From Lemma 5.1 of [2], the product is given by

$$H^{i+j}([G\times G/G\times G]) \xrightarrow{\Delta^*} H^{i+j}([G\times G/G]) \xrightarrow{m_!} H^{i+j-d}([G/G]),$$

where d is the dimension of G. Translating this product to homotopy orbit spaces gives

$$H^{i+j}(G_{hG}\times G_{hG}) \xrightarrow{\Delta^*} H^{i+j}((G\times G)_{hG}) \xrightarrow{m_!} H^{i+j-d}(G_{hG}),$$

which is clearly the product given by applying H^* and Thom isomorphisms to the coproduct on $LBG^{-\mathrm{ad}}$. □

PROPOSITION 8.2. *There is a nonunital ring spectrum structure on $LBG^{+\mathrm{ad}}$ which realizes the coproduct on $H^*([G/G])$ defined in* [2].

REMARK 8.3. However, BGNX have shown that this coproduct is trivial on $H^*([G/G];\mathbb{R})$. It is likely to be nontrivial in any cohomology theory which detects the G-transfer map $S^0[BG^{\mathrm{ad}}]\to S^0$ (such as the orthogonal K-theory when $G=S^1$).

Proof. The diagonal embedding $G\hookrightarrow G\times G$ induces a relative transfer map

$$\tau_G^{G\times G} : (S^{\mathfrak{g}\times\mathfrak{g}}\wedge(G\times G)_+)_{hG\times G} \longrightarrow (S^{\mathfrak{g}}\wedge(G\times G)_+)_{hG}.$$

The left-hand side is equivalent to $(S^{\mathfrak{g}}\wedge G_+)_{hG}\wedge(S^{\mathfrak{g}}\wedge G_+)_{hG}$. Group multiplication in G induces

$$m : (S^{\mathfrak{g}}\wedge(G\times G)_+)_{hG} \longrightarrow (S^{\mathfrak{g}}\wedge G_+)_{hG},$$

since it is G-equivariant. Hence we can define the multiplication on

$$LBG^{\mathrm{ad}} = (S^{\mathfrak{g}}\wedge G_+)_{hG}$$

to be

$$m\circ\tau_G^{G\times G} : LBG^{\mathrm{ad}}\wedge LBG^{\mathrm{ad}} \longrightarrow LBG^{\mathrm{ad}}.$$

This product is the same as the ring structure on LBG^{ad} described in [22] coming from the first term of the transfer operad G_{bG}. It is associative but not unital. Applying cohomology and Thom isomorphisms yields

$$H^i([G/G]) \xrightarrow{m^*} H^i([G\times G/G]) \xrightarrow{\tau^*} H^{i-d}([G\times G/G\times G]) \cong \bigoplus_{r+s=i-d} H^r([G/G])\otimes H^s([G/G]),$$

which is the same as the description of the coproduct in Lemma 5.1 of [2]. □

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