Weeks 4: Symmetric products and the Fox-Neuwirth cell decomposition

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Symmetric products

Definition 1. For a space X and nonnegative integer n, we define $\mathrm{Sym}^n(X)$ or $\mathrm{SP}^n(X)$ to be the n-fold symmetric product of X:

$$\mathrm{Sym}^n(X) = X^{\times n}/S_n,$$

where S_n acts via permutation of coordinates.

The configuration space $\mathrm{Conf}_n(X) \subseteq \mathrm{Sym}^n(X)$ embeds as the open subset of n-tuples of distinct elements. We note that, in contrast to configuration spaces, symmetric products are functorial for all continuous maps (not just injective ones). Consequently, the homotopy type of $\mathrm{Sym}^n(X)$ depends only upon the homotopy type of X. In particular, if X is contractible, $\mathrm{Sym}^n(X) \simeq *$ is, too.

In what is to come, it will be helpful to have the stronger identification of these spaces up to homeomorphism in certain cases.

Proposition 2. *The following hold for* $\mathbb{R}$*:*

1. *The configuration space* $\mathrm{Conf}_n(\mathbb{R})$ *is homeomorphic to the interior of the closed* n*-dimensional simplex,* Δ^n*. Note that this in turn is homeomorphic to* $\mathbb{R}^n$*.*
2. *Similarly,* $\mathrm{Sym}^n(\mathbb{R})$ *is homeomorphic to the complement of a two faces in* Δ^n*.*

Proof. Using the homeomorphism $\mathbb{R} \cong (0,1)$, we of course have $\mathrm{Conf}_n(\mathbb{R}) \cong \mathrm{Conf}_n((0,1))$. Any n-tuple of distinct points $(y_1, \ldots, y_n)$ in $(0,1)$ has a unique reordering $(y_{\sigma(1)}, \ldots, y_{\sigma(n)})$ with the property that $y_{\sigma(i)} < y_{\sigma(i+1)}$. So:

$$\mathrm{Conf}_n(\mathbb{R}) \cong \{(x_1, \ldots, x_n) \mid 0 < x_1 < x_2 < \cdots < x_n < 1\}.$$

We recall that $\Delta^n = \{(z_0, \ldots, z_n) \in \mathbb{R}^n \mid \sum z_i = 1, z_j \geq 0\}$. A homeomorphism from $\mathrm{Conf}_n((0,1))$ to the interior of Δ^n is given by the map

$$(x_1, \ldots, x_n) \mapsto (x_1, x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1}, 1 - x_n)$$

Since none of the x_i are equal to each other, or to 0 or 1, the image consists of elements $(z_0, \ldots, z_n)$ where none of the z_i are equal to 0 or 1; this is precisely the interior of the simplex.

The same map gives a homeomorphism of $\mathrm{Sym}^n(\mathbb{R})$ onto the subspace of Δ^n consisting of points $(z_0, \ldots, z_n)$ where neither z_0 nor z_n are equal to 0 or 1. We recall that Δ^n has $n+1$ faces, all of which are defined by a single coordinate equalling 0. If any of the z_i are equal to 1, then all of the other z_j are 0; this is a single vertex and contained in one of the faces. Thus $\mathrm{Sym}^n(\mathbb{R})$ is the complement of two faces in Δ^n.

□

In contrast, the symmetric product of $\mathbb{C}$ admits a much simpler description:

Proposition 3. $\mathrm{Sym}^n(\mathbb{C})$ *is homeomorphic to* $\mathbb{C}^n$.

Proof. Let Poly_n denote the space of monic, degree n polynomials over $\mathbb{C}$;

$$\mathrm{Poly}_n = \{f(z) = z^n + a_1 z^{n-1} + \ldots a_{n-1} z + a_n\} \cong \mathbb{C}^n.$$

There is a homeomorphism $\mathrm{Poly}_n \to \mathrm{Sym}^n \mathbb{C}$ which carries f to the unordered n-tuple of its (not necessarily distinct) roots. That this is a bijection is a consequence of the fundamental theorem of algebra.

□

Note that for an element $\underline{z} = (z_1, \ldots, z_n) \in \mathrm{Sym}^n \mathbb{C}$, the monic polynomial f with roots at $\underline{z}$ has coefficients $a_i = (-1)^i e_i(z_1, \ldots z_n)$, where e_i is the i^{th} elementary symmetric polynomial. Thus an explicit set of coordinates on $\mathrm{Sym}^n(\mathbb{C})$ is given by the elementary symmetric polynomials.

Finally, we have:

Proposition 4. $\mathrm{Sym}^n(\mathbb{C}P^1)$ *is homeomorphic to* $\mathbb{C}P^n$.

Proof. Define

$$\mathrm{Homog}_n := \{f(z,w) = a_0 z^n + a_1 z^{n-1} w + \cdots + a_{n-1} z w^{n-1} + a_n w^n \mid (a_0, \ldots, a_n) \neq 0\}$$

to be the space of nonzero homogenous polynomials of degree n in two variables z, w; it is homeomorphic to $\mathbb{C}^{n+1} \setminus \{0\}$. Letting $\mathbb{C}^\times$ act by scaling the coefficients of such a polynomial, $\mathrm{Homog}_n / \mathbb{C}^\times \cong \mathbb{C}P^n$.

Define a homeomorphism $\mathrm{Sym}^n(\mathbb{C}P^1) \to \mathrm{Homog}_n / \mathbb{C}^\times$ by

$$(z_1, \ldots z_n) \mapsto f(z,w) := \prod_{i=1}^{n} (z - z_i w).$$

Here, if $z_i = \infty$, we interpret the factor $z - z_i w$ as $-w$. An inverse is given as follows: factor $f(z,w) = w^m g(z,w)$ for some m so that $g(z,w)$ is indivisible by w. Then map f to the n-tuple consisting of m points at ∞, along with the $n-m$ roots in $\mathbb{C}$ of $g(z,1)$ (which is of degree $n-m$).

□

The Fox-Neuwirth cell decomposition of $\mathrm{Conf}_n(\mathbb{C})$

We will describe[1] the results of [FN62] (see also [GS12]) which give a decomposition of $\mathrm{Conf}_n(\mathbb{R}^m)$ into spaces homeomorphic to Euclidean spaces. This does not give a cell-decomposition of $\mathrm{Conf}_n(\mathbb{R}^m)$, but rather of its 1-point compactification. We will restrict our focus to the case $m = 2$ (i.e., $\mathrm{Conf}_n(\mathbb{C})$), and encourage the interested reader to extend these constructions to $m > 2$.

An ***ordered*** partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n has $\sum \lambda_i = n$. With apologies to the combinatorialists present, we will write $k = |\lambda|$ for the number of ***parts*** of λ. The n^{th} symmetric product $\mathrm{Sym}_n(\mathbb{R})$ of the real line has a stratification by these partitions:

$$\mathrm{Sym}_n(\mathbb{R}) = \coprod_{\lambda \vdash n} \mathrm{Sym}_\lambda(\mathbb{R})$$

where elements of $\mathrm{Sym}_\lambda(\mathbb{R})$ consist of $|\lambda|$ distinct points $x_1, \ldots x_{|\lambda|}$, the i^{th} of which has multiplicity λ_i. Further, since $\mathbb{R}$ is ordered, we insist that $x_1 < \cdots < x_{|\lambda|}$. This space is evidently homeomorphic to $\mathrm{Conf}_{|\lambda|}(\mathbb{R})$, which is in turn homeomorphic to $\mathbb{R}^{|\lambda|}$, as shown above.

Define a map $\pi : \mathrm{Conf}_n(\mathbb{C}) \to \mathrm{Sym}_n(\mathbb{R})$ by taking real parts:

$$\pi(z_1, \ldots, z_n) = (\Re(z_1), \ldots, \Re(z_n)),$$

and let $\mathrm{Conf}_\lambda(\mathbb{C})$ denote the preimage of $\mathrm{Sym}_\lambda(\mathbb{R})$ under π. This subspace is homeomorphic to

$$\mathrm{Sym}_\lambda(\mathbb{R}) \times \prod_{i=1}^{|\lambda|} \mathrm{Conf}_{\lambda_i}(\mathbb{R}),$$

[1]This discussion is taken from a forthcoming paper which is joint with TriThang Tran; the results described are not original to us, although the presentation is. I suppose that I don't feel too much shame in plagiarizing myself when summarizing someone else's work.

where the configuration factors record the imaginary part of the configuration of λ_i points lying over the i^{th} term in the set of the real coordinates of $\underline{z}$. We again employ the fact that $\mathrm{Conf}_k(\mathbb{R}) \cong \mathbb{R}^k$ and conclude that $\mathrm{Conf}_\lambda(\mathbb{C}) \cong \mathbb{R}^{n+|\lambda|}$.

Proposition 5. *The collection of subspaces* $\mathrm{Conf}_\lambda(\mathbb{C})$ *forms a cellular decomposition of the 1-point compactification* $\mathrm{Conf}_n(\mathbb{C}) \cup \{\infty\}$*; the cells of dimension* d *are indexed by those partitions* λ *with* $n + |\lambda| = d$*. Furthermore, the closure of the cell* $\mathrm{Conf}_\lambda(\mathbb{C})$ *is the union*

$$\overline{\mathrm{Conf}_\lambda(\mathbb{C})} = \coprod_\rho \mathrm{Conf}_\rho(\mathbb{C})$$

over ordered partitions ρ *which are refined by* λ.

We must explain the last comment. Loosely speaking, the boundaries of the cell described above occur in two ways. First, points in a configuration may approach each other or $\pm i\infty$ along vertical lines (in which case their boundary is the point at infinity). Secondly, the i^{th} and $i+1^{\text{st}}$ vertical columns of configurations may approach each other horizontally, in which case the associated component of the boundary is given in terms of the cell $\mathrm{Conf}_\rho(\mathbb{C})$, where ρ is obtained from λ by summing λ_i and λ_{i+1}.

It is worth noting that, other than $\{\infty\}$, there are no cells of dimension less than or equal to n in this decomposition. Therefore, we have

Corollary 6. *For* $* \geq n$,

$$H_* \mathrm{Conf}_n(\mathbb{C}) \cong H_c^{2n-*} \mathrm{Conf}_n(\mathbb{C}) = H^{2n-*}(\mathrm{Conf}_n(\mathbb{C}) \cup \{\infty\}, \{\infty\}) = 0.$$

There are more computational proofs of this fact that rely on our previous computation of the homology of $\mathrm{PConf}_n(\mathbb{C})$, though perhaps none quite as enlightening. In any case, this follows from the cellular decomposition and the following exercise:

Exercise 7. $\mathrm{Conf}_n(\mathbb{C})$ is a $2n$-dimensional, oriented, non-compact manifold.

A word of warning: there are 2^{n-1} ordered partitions of n, so this is not the most efficient approach to computing $H_* \mathrm{Conf}_n(\mathbb{C})$ as n grows.

References

[FN62] R. Fox and L. Neuwirth, *The braid groups*, Math. Scand. **10** (1962), 119–126. MR 0150755 (27 #742)

[GS12] Chad Giusti and Dev Sinha, *Fox-Neuwirth cell structures and the cohomology of symmetric groups*, Configuration spaces, CRM Series, vol. 14, Ed. Norm., Pisa, 2012, pp. 273–298. MR 3203643