some solutions to miscellaneous complex analysis prelim problems*

David Morawski

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Problem (Spring 2012, #3)**.** Let $f(z)$ be an entire holomorphic function. Suppose that there are positive real numbers a, b, and k such that $|f(z)| \leq a + b|z|^k$ for all $z \in \mathbb{C}$. Prove that f is a polynomial.

Proof. Since $f(z)$ is entire, it assumes the form of its Taylor expansion about zero for all $z \in \mathbb{C}$. That is,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

for all z. Let $R > 0$ be given. The Cauchy Estimate gives

$$|f^{(n)}(0)| \leq \frac{n!}{R^n}(a + b|z|^k)$$

for $|z| = R$. So, letting $R \to \infty$, $f^{(n)}(0) = 0$ whenever $n > k$. So f is a polynomial. □

Problem (Spring 2012, #4)**.** Prove that there is no function f that is analytic on the punctured disk $D = \{z \in \mathbb{C} : 0 < |z| < 1\}$, and f' has a simple pole at 0.

Proof. Suppose such a function f exists. Because f is analytic in D, so is f'. So f' has a Laurent decomposition in D and because f' has a simple pole at zero, it assumes the form

$$f'(z) = \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \cdots$$

for $0 < |z| < 1$. Integrating these expression gives us

$$f(z) = a_{-1} \log z + c + a_0 z + \frac{a_1}{2} z^2 + \cdots ,$$

where c is some constant. But there is no branch of $\log z$ that is analytic in D, so this is a contradiction. □

*These solutions have not been checked by any prelim graders.

Problem (Spring 2012, #5). Suppose that an entire holomorphic function g satisfies $g(1-z) = 1 - g(z)$ for all $z \in \mathbb{C}$. Show that $g(\mathbb{C}) = \mathbb{C}$.

Proof. The problem, as stated, is wrong. The function $g(z) = 1/2$ is a counterexample. Adding the hypothesis that g is nonconstant, however, we may apply Picard's Little Theorem to conclude that g misses at most one point in the complex plane. Thus, $g(\mathbb{C}) \supseteq \mathbb{C} - \{z_0\}$ for some z_0. Note that $1 - z_0$ lies in the image of $\mathbb{C}$ under g, say with $g(z_1) = 1 - z_0$. Then $g(1 - z_1) = 1 - g(z_1) = 1 - (1 - z_0) = z_0$. So $g(\mathbb{C}) = \mathbb{C}$. $\square$

Problem (Spring 2012, #6). Construct an analytic, one-to-one, onto map from $\{z : |z| < 1, \text{Re}(z) > 0\}$ to $\{z : |z| < 1\}$.

Proof. The task is to construct a conformal map from the right half-disc to the open unit disc, which we will do by composing several conformal maps. First, the map $f(z) = e^{\frac{\pi}{2}i}z = iz$ takes the right half-disc to the upper half-disc.

Now let's look at the map $z \mapsto w = \frac{1+z}{1-z}$. It's not hard to show that if z is in the upper half-disc, then w satisfies $\text{Re}(w) > 0$ and $\text{Im}(w) > 0$ – i.e., w lies in the first quadrant. One can calculate the inverse of this map and check that the image lies in the upper half disc, thus showing that $z \mapsto w$ gives a conformal mapping between the upper half-disc and the first quadrant.

Next we compose with $z \mapsto z^2$, which takes the first quadrant to the upper half plane. Then we compose with $z \mapsto \frac{z-i}{z+i}$. We can prove that this is an isomorphism between the upper half plane and the unit disc by using the method described in the previous paragraph. $\square$

Problem (Spring 2012, #7). Let $h : \mathbb{C} \to \mathbb{R}$ be a non-constant harmonic function. Show that h has at least one zero.

Proof. Suppose h has no zeros. Without loss of generality, assume h is positive. Now let g be a harmonic conjugate of h, so $f = h + ig$ is entire. Then e^{-f} is bounded, since $|e^{-f}| = |e^{-h}| < 1$ since $h > 0$ (in the case that $h < 0$, we consider e^f). So e^{-f} is constant, which implies that f is constant. Then h must also be constant – a contradiction. $\square$

Problem (Fall 2011, #2). Let h be a nowhere zero, entire function. Prove that there exists an entire function g such that $e^g = h$.

Proof. Since h is nowhere zero and entire, h'/h is entire. Thus, there is an entire function g such that $g' = h'/h$ (one way to see this is to integrate the Taylor series expansion about zero of h'/h). Then fe^{-g} has derivative equal to zero, thus $f = e^{-g}k$ for a constant k. Incorporating this constant into $-g$ finishes the exercise. $\square$

Problem (Fall 2011, #4). Let f be a holomorphic function in the right half-plane $\{z \in \mathbb{C} : \text{Re}(z) > 0\}$. Suppose that $|f(z)| < 1$ for all z in the domain of f, and $f(1) = 0$. Find the largest possible value of $|f(2)|$.

Proof. We will construct a fractional linear transformation g from the unit disc $\mathbb{D}$ to the right half plane H such that $g(0) = 1$. Once we have such a g, the composition $f \circ g$ will satisfy the hypotheses of Schwarz Lemma. Then $|f(2)| = |f(g(g^{-1}(2)))|$ will be at most $|g^{-1}(2)|$.

The map $z \mapsto \frac{z-i}{z+i}$ maps the upper half plane conformally onto $\mathbb{D}$. Thus, its inverse, which turns out to be $z \mapsto i\frac{1+z}{1-z}$, is an analytic bijection taking $\mathbb{D}$ to the upper half plane. The map by $z \mapsto -iz$ conformally maps the upper half plane to the right half plane, so the composition

$$g(z) = \frac{1+z}{1-z}$$

is a conformal mapping from $\mathbb{D}$ to H. We calculate $g^{-1}(2) = 1/3$, so $|f(2)| \leq 1/3$. □

Problem (Fall 2011, #5)**.** Let $f(z)$ be an entire function. Suppose that $f(z) = f(z+1)$ and $|f(z)| \leq e^{|z|}$ for all $z \in \mathbb{C}$. Prove that $f(z)$ must be constant.

Proof. □

Problem (Fall 2011, #7)**.** Prove that there is no one-to-one conformal map of the punctured disc $\mathbb{D} - \{0\} = \{z : 0 < |z| < 1\}$ onto the annulus $A = \{z : 1 < |z| < 2\}$.

Proof. Suppose f is such a map. Since f is bounded near zero, Riemann's Theorem on Removable Singularities [Gamelin 172] tells us that zero is removable. So f can be extended analytically to the unit disc. Let $w_0 = f(0)$. Then, by the continuity of f, w_0 lies in the closure of the annulus $\overline{A}$.

Suppose w_0 lies on the boundary of the annulus ∂A. Then $f(\mathbb{D}) = A \cup \{w_0\}$, which contradicts the Open Mapping Theorem for Analytic Functions.

So w_0 lies in A. Since f surjects the punctured disc on the annulus, there is a point $z_0 \in \mathbb{D} - \{0\}$ such that $f(z_0) = w_0$. Let V and W be disjoint neighborhoods of z_0 and zero, respectively. Then, again by the Open Mapping Theorem, $f(V)$ and $f(W)$ are open. Furthermore, they both contain w_0, hence are non-empty. Let w_1 be a point in their intersection that is distinct from w_0. There are distinct points $z_1 \in V$ and $z_2 \in W$ such that $f(z_1) = w_1 = f(z_2)$, contradicting that f is injective. Thus, no such map f can exist. □