The relaxed game chromatic index of trees

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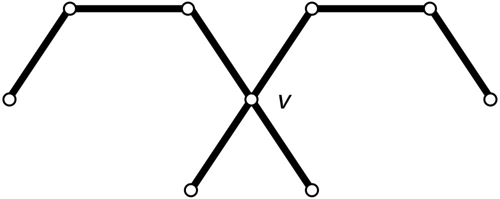
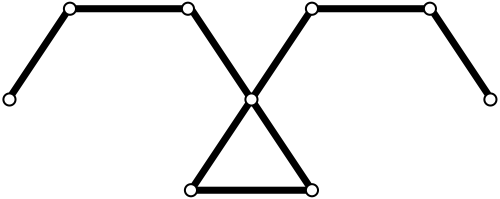
Day one of graph theory

A graph $G = (V, E)$: A tree $T = (V, E)$ (no cycles):



Not a tree (≥ 1 cycle):



Degree of a vertex v: the number of vertices adjacent to v.

$\Delta(T)$ = maximum degree of a tree T.

Edge coloring

A **proper edge coloring** of a graph G is an assignment of colors to each edge of G such that if e and f are adjacent edges (i.e., share a common vertex), then e and f are assigned different colors. For example:

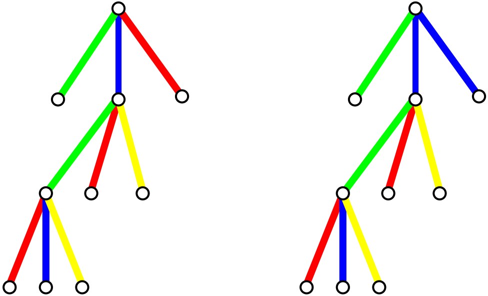


Figure: Proper and improper colorings (left to right).

The minimum number of colors needed to properly color the edges of a graph G is called the **chromatic index** of G, denoted $\chi'(G)$.

Edge coloring bounds

It's not difficult to see that $\Delta(G) \leq \chi'(G)$ for any graph G.

Less trivially, we have the following upper bound for the chromatic index:

Theorem (Vizing, 1964)

For any graph G, $\chi'(G) \leq \Delta(G) + 1$.

Thus, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ for any graph G.

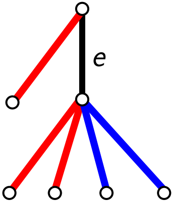
Unfortunately, it's NP-complete to determine whether the chromatic index of a graph is Δ or $\Delta + 1$ (Holyer, 1981).

Relaxed edge colorings

A d-**relaxed edge coloring** is an edge coloring in which we allow an edge e to be colored α if:

- e is adjacent to at most d α-colored edges, and
- if f is an α-colored edge adjacent to e, then f is adjacent to at most $d-1$ α-colored edges.

For example:



In a 2-relaxed edge coloring, e could be colored with blue, but *not* red.

The relaxed edge coloring game

The (r, d)-relaxed edge coloring game on a graph G is as follows:

- A two player game—say Alice and Bob, where Alice plays first.
- Alice and Bob alternate coloring one uncolored edge per turn.
- r colors available.
- The players agree to a d-relaxed proper coloring.
- If the graph is properly colored, Alice wins.
- In the case of an uncolorable edge, Bob wins.

The relaxed game chromatic index and edge-game defect

- For a fixed defect d, the d-**relaxed game chromatic index** of a graph G, denoted ${}^{d}\chi'_{\mathrm{g}}(G)$, is the least number of colors r such that Alice has a winning strategy in the (r, d)-relaxed edge coloring game.
- We write the 0-relaxed edge-game chromatic index as $\chi'_{\mathrm{g}}(G)$.
- For a fixed number of colors r, the r-**edge-game defect** of a graph G, denoted $\mathrm{def}'_{\mathrm{g}}(G, r)$, is the least defect d such that Alice has a winning strategy in the game.

What's already known ...

Theorem (Cai & Zhu, 2001)

For any tree T, $\chi'_g(T) \leq \Delta(T) + 2$.

Theorem (Dunn, 2007)

For any tree T with $\Delta(T) = \Delta$, $\operatorname{def}'_g(T, \Delta + 1) \leq 1$. Furthermore, if $d \geq 1$, then ${}^d\chi'_g(T) \leq \Delta + 1$.

Theorem (Dunn, 2007)

For any tree T with $\Delta(T) = \Delta$, $\operatorname{def}'_g(T, \Delta) \leq 3$. Furthermore, if $d \geq 3$, then ${}^d\chi'_g(T) \leq \Delta$.

Can we keep going? ... yes.

The main result

Theorem (Dunn, M., Nordstrom, 2009)

Let T be a tree with $\Delta(T) = \Delta$. Then, for any $k = 1, 2, \ldots, \Delta - 1$, $\mathrm{def}'_g(T, \Delta - k) \leq 2k + 2$. Furthermore, if $d \geq 2k + 2$, then ${}^d\chi'_g(T) \leq \Delta - k$.

Summary of game chromatic index results

Theorem (Cai & Zhu, 2001)

For any tree T, $\chi'_g(T) \leq \Delta(T) + 2$.

Theorem (Dunn, 2007)

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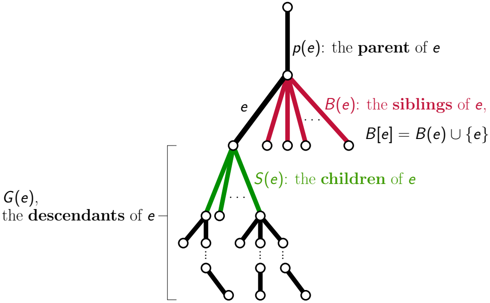
Theorem (Dunn, M., Nordstrom, 2009)

Let T be a tree with $\Delta(T) = \Delta$. Then, for any $k = 1, 2, \ldots, \Delta - 1$, $\mathrm{def}'_g(T, \Delta - k) \leq 2k + 2$. Furthermore, if $d \geq 2k + 2$, then ${}^d\chi'_g(T) \leq \Delta - k$.

Some definitions



For a colored edge e, we let $c(e)$ denote the color of e.

For an edge e, the **defect** of e, denoted $\operatorname{def}(e)$, is the number of edges adjacent to e having the same color as e. If e is uncolored, $\operatorname{def}(e) = 0$.

A color α is **eligible** for an edge e if the parent of e is not colored α.

Alice only uses eligible colors.

Conventions for the rest of this talk

We fix a tree $T = (V, E)$ with $\Delta(T) = \Delta$.

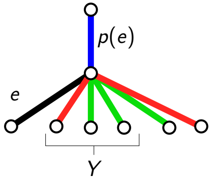
We also fix $k = 1, 2, \ldots, \Delta - 1$.

Recall, our result tells us a sufficient defect when Alice and Bob are playing the game with $\Delta - k$ colors.

Definition: secure

For an edge e, we say that $B[e]$ is **secure** if there exists a colored $Y \subseteq B[e]$ such that $|Y| \geq k$ and, for every edge $f \in Y$, there exists $f' \in B[e] - Y$ with $c(f) = c(f')$.

For example:



Here we have $\Delta = 7$ and $k = 3$. If Alice and Bob are playing with $\Delta - k = 4$ colors, then there is an eligible color for e that does not appear among the siblings of e.

This is always the case when $B[e]$ is secure, for any edge e.

Alice's strategy: an overview

Alice's strategy consists of a

- **Search stage:** Alice locates the edge e that she will color, in response to the move just made by Bob.
- **Coloring stage:** Alice chooses an eligible color for e, with the hope of minimizing the defect of nearby edges.

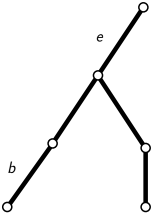
Alice's strategy: activation

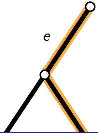
Alice will maintain a set $A \subseteq E$ of **active** edges.

Once an edge is *activated*–i.e., put into A–it remains active for the remainder of the game.

All colored edges are active.

Alice activates edges in response to moves made by Bob.

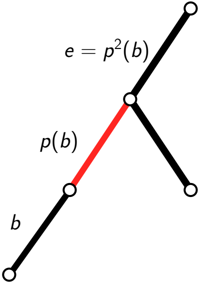


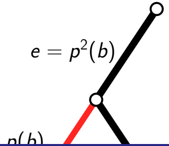


Alice's strategy: searching

Suppose Bob has just colored an edge b the same color as its parent $p(b)$.

1. If the grandparent $p^2(b)$ is uncolored, Alice selects it and moves to the coloring stage.
2. Otherwise, Alice selects any uncolored sibling of $p(b)$.

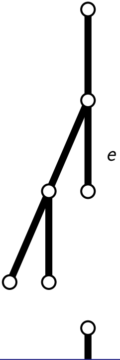




Alice's strategy: coloring

Suppose Alice is about to color an edge e:

- If $B[e]$ is secure, then Alice chooses an eligible color for e that does not appear among the siblings of e.
- If possible, let f be the last edge to be colored with an eligible color for e, such that $p(f)$ is a sibling of e and $c(f) = c(p(f))$. If such an edge exists, Alice colors e with $c(f)$.
- Otherwise, Alice chooses an eligible color for e that minimizes $\text{def}(e)$.



Conclusion

Given any tree T with $\Delta(T) = \Delta, k = 1, 2, \ldots, \Delta - 1$, and considering the relaxed edge coloring game with $\Delta - k$ colors:

We have outlined a strategy for Alice such that she may always make a legal move, while keeping the defect of any edge at most $2k + 2$.

Furthermore, if the defect is greater than $2k + 2$, the arguments remain valid.

Since Bob may adopt Alice's strategy at any point during the game, both players always have a legal move that respects the defect. Thus,

- $\mathrm{def}'_{\mathrm{g}}(T, \Delta - k) \leq 2k + 2$ and,
- ${}^{d}\chi'_{\mathrm{g}}(T) \leq \Delta - k$, whenever $d \geq 2k + 2$.

Thanks!

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